Unit V

Line Integrals

The basic theme here is that complex line integrals will mirror much of what we've seen for multivariable calculus line integrals. But, just like working with $e^{i\theta}$ is easier than working with sine and cosine, complex line integrals are easier to work with than their multivariable analogs. At the same time they will give deep insight into the workings of these integrals.

- The complex plane: $z = x + iy$
- The complex differential $dz = dx + idy$
- A curve in the complex plane: $\gamma(t) = x(t) + iy(t)$, defined for $a \le t \le b$.
- A complex function: $f(z) = u(x, y) + iv(x, y)$

Note. Line integrals are also called path or contour integrals.

Given the ingredients we define the complex line integral $\int_\gamma f(z)\,dz$ by

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \quad (1)$$

You should note that this notation looks just like integrals of a real variable. We don't need the vectors and dot products of line integrals in $\mathbf{R}^2$. Also, make sure you understand that the product $f(\gamma(t))\gamma'(t)$ is just a product of complex numbers.

An alternative notation uses $dz = dx + idy$ to write

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + idy) \quad (2)$$

Let's check that Equations 1 and 2 are the same. Equation 2 is really a multivariable calculus expression, so thinking of $\gamma(t)$ as $(x(t), y(t))$ it becomes

$$\int_\gamma f(z)\,dz = \int_a^b (u(x(t), y(t)) + iv(x(t), y(t))(x'(t) + iy'(t))dt$$

$$\int_\gamma f(z)\,dz = \int_a^b (u(x(t), y(t)) + iv(x(t), y(t))(x'(t) + iy'(t))dt$$

But, $u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$ and $x'(t) + iy'(t) = \gamma'(t)$ so the right hand side of this equation is

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

That is, it is exactly the same as the expression in Equation 1.

Example 3.1. Compute $\int_\gamma z^2 dz$ along the straight line from 0 to $1 + i$.

<u>answer:</u> We parametrize the curve as $\gamma(t) = t(1 + i)$ with $0 \le t \le 1$. So $\gamma'(t) = 1 + i$. The line integral is

$$\int z^2 dz = \int_0^1 t^2(1 + i)^2(1 + i)\,dt = \frac{2i(1 + i)}{3}.$$

Unit V

Example 3.2. Compute $\int_\gamma \bar{z}\, dz$ along the straight line from 0 to $1+i$.

answer: We can use the same parametrization as in the previous example. So,

$$\int_\gamma \bar{z} dz = \int_0^1 t(1-i)(1+i)\, dt = 1.$$

Example 3.3. Compute $\int_\gamma z^2\, dz$ along the unit circle.

answer: We parametrize the unit circle by $\gamma(\theta) = e^{i\theta}$, where $0 \le \theta \le 2\pi$. We have $\gamma'(\theta) = ie^{i\theta}$. So, the integral becomes

$$\int_\gamma z^2\, dz = \int_0^{2\pi} e^{2i\theta} ie^{i\theta}\, d\theta = \int_0^{2\pi} ie^{3i\theta}\, d\theta = \frac{e^{i3\theta}}{3}\Big|_0^{2\pi} = 0.$$

Example 3.4. Compute $\int \bar{z} dz$ along the unit circle.

answer: Parametrize C: $\gamma(t) = e^{it}$, with $0 < t < 2\pi$. So, $\gamma'(t) = ie^{it}$. Putting this into the integral gives

$$\int_C \bar{z}\, dz = \int_0^{2\pi} \overline{e^{it}}\, i e^{it}\, dt = \int_0^{2\pi} i\, dt = \boxed{2\pi i.}$$

Cauchy Integral Theorem

Let f be holomorphic inside Ω be bounded by a closed piecewise-smooth simple curve γ and also at the point of γ. Then

$$\oint_\gamma f(z)\, \mathrm{d}z = 0$$

Proof. Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, so

$$\oint_\gamma f(z)\, \mathrm{d}z = \oint_\gamma (u+iv)(\mathrm{d}x + i\,\mathrm{d}y)$$
$$= \oint_\gamma u\, \mathrm{d}x - v\, \mathrm{d}y + i \oint_\gamma v\, \mathrm{d}x + u\, \mathrm{d}y$$

and by Green's formula

$$= \iint_\Omega (-v'_x - u'_y)\, \mathrm{d}x\, \mathrm{d}y + i \iint_\Omega (u'_x - v'_y)\, \mathrm{d}x\, \mathrm{d}y = 0$$

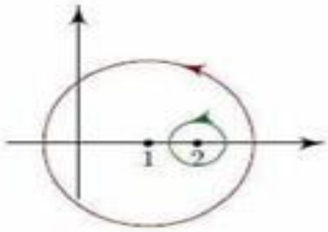
Deformation

Let γ_1 and γ_2 be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1 and suppose f is holomorphic in a domain containing the region between γ_1 and γ_2. Then

$$\oint_{\gamma_1} f(z)\, \mathrm{d}z = \oint_{\gamma_2} f(z)\, \mathrm{d}z$$

Example Take $\gamma = \{z \in \mathbb{C} : |z-1| = 2\}$

$$\oint_\gamma \frac{1}{z^2-4}\,dz = \oint_\gamma \frac{1}{(z-2)(z+2)}\,dz = \frac{1}{4}\oint_\gamma \left(\frac{1}{z-2} - \frac{1}{z+2}\right) dz$$



Since $\dfrac{1}{z+2}$ is holomorphic inside γ and on γ, by Cauchy Goursat:

$$\oint_\gamma \frac{1}{z+2}\,dz = 0$$

Now

$$\oint_\gamma \frac{1}{z-2}\,dz = \oint_{z:|z-2|=\frac{1}{2}} \frac{1}{z-2}\,dz$$

Changing variables $z - 2 = w$

$$= \oint_{|w|=\frac{1}{2}} \frac{1}{w}\,dw$$

Parametrising $w = \frac{1}{2}e^{it}$, $t \in [0, 2\pi]$

$$= \int_0^{2\pi} \frac{1}{\frac{1}{2}e^{it}} \cdot \frac{1}{2} i e^{it}\,dt = 2\pi i$$

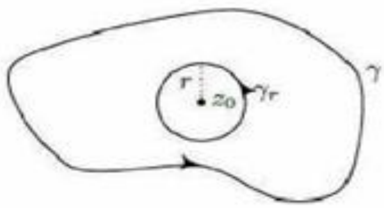
Hence

$$\oint_\gamma \frac{1}{z^2-4}\,dz = i\frac{\pi}{2}$$

Cauchy integral Formula

Let f be holomorphic inside and on a simple, closed piecewise-smooth curve γ. Then for any point z_0 interior to γ, we have

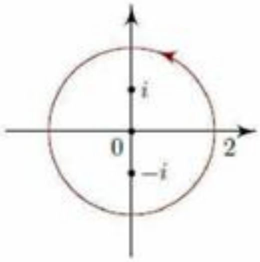
$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-z_0}\,dz$$



Unit V

Example Calculate

$$I = \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z-i)(z+i)}\, dz$$



Using partial fractions, we get

$$I = \frac{1}{2\pi i} \cdot \frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z-i} - \frac{e^z}{z+i} \right) dz$$

By Cauchy's integral formula:

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{z-i}\, dz = e^i$$

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{z+i}\, dz = e^{-i}$$

And so

$$I = \frac{1}{2i}(e^i - e^{-i}) = \sin 1$$

Generalized Cauchy Formula

Let f be holomorphic in an open set Ω. Then f has infinitely many complex derivatives in Ω. Moreover for a simple, closed piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{d^n}{dz^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\, d\eta$$

and so

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \to \frac{(n-1)!}{2\pi i} \oint_\gamma f(z) \frac{1}{(\eta - z)^2} \cdot \frac{n}{(\eta - z)^{(n-1)}}\, d\eta$$

$$= \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\, d\eta$$

Complex Power Series

Definition. A complex power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

where $z_0 \in \mathbb{C}$ and a_n are complex constants.

Example

$$\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{n!}$$

In this case,

$$L = \lim_{n \to \infty} \left| \frac{(z - z_0)^{n+1}}{(n+1)!} \cdot \frac{n!}{(z - z_0)^n} \right|$$

$$= \lim_{n \to \infty} \frac{|z - z_0|}{n+1} = 0$$

Therefore $\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}$ converges for all $z \in \mathbb{C}$.

Unit V

Definition (Radius of convergence). There is a real number $R \in \{[0, \infty]\}$ such that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z : |z-z_0| < R$ and diverges for all $z : |z-z_0| > R$ and may or may not converge for $z : |z-z_0| = R$. Such value of R is called the radius of convergence for the power series.

Remark. If $R = 0$ then the power series converges only at $z = z_0$. If $R = \infty$ the power series converges absolutely for all $z \in \mathbb{C}$.

For many power series the radius of convergence can be calculated according to

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad (**)$$

or

$$R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$$

provided either limit exists or is equal to infinity. Indeed, let us check (**): we know that the series converges if

$$L = \lim_{n\to\infty} \frac{|a_{n+1}(z-z_0)^{n+1}|}{a_n(z-z_0)^n|} = |z-z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

This implies that if

$$|z-z_0| < \lim_{n\to\infty} |a_n|/|a_{n+1}|$$

then the power series converges.

Example Find R for $\sum_{n=1}^{\infty} n^2(z-z_0)^n$.

$$R = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

$\Longrightarrow \sum_{n=1}^{\infty} n^2(z-z_0)^n$ is absolutely convergent for all $z : |z-z_0| < 1$.

Taylor Expansion

Let f be holomorphic in an open set Ω and let $z_0 \in \Omega$. Then

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

valid in all circles $\{z \in \mathbb{C} : |z-z_0| < r\} \subset \Omega$.

Definition. The expansion

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

is called the Taylor series of f about z_0. The special case in which $z_0 = 0$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

this series is called the Maclaurin series for f.

Examples

(i) $f(z) = e^z; z_0 = 0 \Longrightarrow f^{(n)}|_{z=0} = 1$. So

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^m, \quad R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty$$

(ii) $$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1 \quad \text{Then } R = 1.$$

Unit V

(iii) $\text{Log}(1-z)$. Note that

$$(\text{Log}(1-z))' = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

Integrating both sides we get

$$\text{Log}\,(1-z) = -\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} + C = -\sum_{n=1}^{\infty} \frac{1}{n} z^n + C$$

where $C = \text{Log}\,(1-0) = 0$.

(iv) $f(z) = \frac{1}{1+z}$ about $z_0 = i$. Then

$$\begin{aligned}\frac{1}{1+z} &= \frac{1}{1+i+z-i}\\ &= \frac{1}{1+i} \cdot \frac{1}{1-(-\frac{z-i}{1+i})}\\ &= \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^n}\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1+i)^{n+1}} (z-i)^n\end{aligned}$$

where R is defined by the inequality

$$\frac{|z-i|}{1+i} < 1 \quad \text{or} \quad |z-i| < \sqrt{2}$$

Laurent Expansion

Let f be holomorphic in the annulus

$$D = \{z : r < |z - z_0| < R\}, \quad r, R > 0$$

Then $f(z)$ can be expressed in the form $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z_0)^{(n+1)}}\, d\eta$$

and whose γ is any simple, closed, piecewise-smooth curve in D that contains z_0 in its interior.

Unit V

Examples

(i) Find the Laurent series at $z_0 = 0$ for $f(z) = 1/(z-1)$ for $z : |z| > 1$.

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{k=1}^{\infty}\frac{1}{z^k}$$

This converges for $|z| > 1$.

(ii) Find the Laurent series at $z_0 = 0$ for $f(z) = 1/z(z+2)$ for $0 < |z| < 2$.

$$\begin{aligned}\frac{1}{z(z+2)} &= \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right) \\ &= \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{4(1+z/2)} \\ &= \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{4}\sum_{n=0}^{\infty}(-z/2)^n \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^n}{2^{n+2}} + \frac{1}{2}\cdot\frac{1}{z}\end{aligned}$$

Singularities

Singularity . A point z_0 is called a *singularity* of a complex function f it f is not holomorphic at z_0, but every neighbourhood of z_0 contains at least one point at which f is holomorphic.

Isolated Singularity . A singularity z_0 of a complex function is said to be *isolated* if there exists a neighbourhood of z_0 in which z_0 is the only singularity of f.

Examples (Singularities).

(i) $f(z) = \frac{z}{1-z}, z_0 = 1$.

(ii) $f(z) = e^{1/z^2}, z_0 = 0$,

(iii) $f(z) = \frac{1}{(z+2)^2}, z_0 = -2$.

Definition (Pole of order m, removable and essential singularity). Suppose a holomorphic function f has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is the Laurent expansion of f valid in some annulus $0 < |z - z_0| < R$. Then

(i) If $a_n = 0$ for all $n < 0$, z_0 is called a *removable singularity*.

(ii) If $a_n = 0$ for $n < -m$ where m is a fixed positive integer, but $a_{-m} \neq 0$, z_0 is called a *pole of order m*.

(iii) If $a_n \neq 0$ for infinitely many negative n's, z_0 is called an *essential singularity*.

Examples (i) $f(z) = \frac{\sin z}{z}$, (ii) $f(z) = \frac{1}{z^3(z+2)^2}$, (iii) $f(z) = e^{1/z}$.

Unit V

Theorem

A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Zeros of holomorphic functions

Definition (Zero of order m). We say that f has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0, k = 0, 1, \ldots m-1$ and $f^{(m)}(z_0) \neq 0$.

Theorem

A holomorphic function f has a zero of order m at z_0 iff it can be written in the form $f(z) = (z-z_0)^m g(z)$ where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Corollary *The zeros of a non-constant holomorphic function are isolated. That is every zero has a neighbourhood inside of which it is the only zero.*

Residue Theory

We have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad 0 < |z-z_0| < R$$

Definition (Residue). $\operatorname{Res}[f, z_0] = a_{-1}$.

Theorem:

Let $\gamma \subset \{z : 0 < |z-z_0| < R\}, R > 0$ be a simple, closed piecewise-smooth curve that contains z_0 and f is holomorphic in $\{z : 0 < |z-z_0| < R\}$. Then

$$\operatorname{Res}[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz$$

Residue

Let f be holomorphic inside and on a simple, closed piecewise-smooth curve γ except at the singularities $z_1, z_2, \ldots, z_n$ in its interior. Then

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}[f, z_j]$$

Example $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots$. Then

$$\oint_{|z|=1} e^{1/z}\, dz = 2\pi i$$

since $a_{-1} = 1$.

Unit V

Calculating Residues

Let

$$f(z) = a_{-m}(z-z_0)^{-m} + \cdots + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0) + \ldots$$

Introduce $g(z) = (z-z_0)^m f(z)$.

For $m = 1$,

$$g(z) = a_{-1} + a_0(z-z_0) + \ldots$$

Then

$$\boxed{\operatorname{Res}[f, z_0] = \lim_{z\to z_0} g(z) = \lim_{z\to z_0} (z-z_0)f(z)}$$

For $m = 2$, then

$$g(z) = a_{-2} + a_{-1}(z-z_0) + a_0(z-z_0)^2 + \ldots$$

$$\operatorname{Res}[f, z_0] = a_{-1} = \frac{\mathrm{d}}{\mathrm{d}z} g(z)\Big|_{z=z_0} = \lim_{z\to z_0} \frac{\mathrm{d}}{\mathrm{d}z}((z-z_0)^2 f(z))$$

Residues for repeated roots: Inductively for m,

$$\boxed{\operatorname{Res}[f, z_0] = \lim_{z\to z_0} \frac{1}{(m-1)!}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}((z-z_0)^m f(z))}$$

Examples

(i) Evaluate

$$\oint_\gamma \frac{1}{z^5 - z^3}\,\mathrm{d}z, \quad \gamma = \{z : |z| = 1/2\}$$

We have

$$\frac{1}{z^5 - z^3} = \frac{1}{z^3}\frac{1}{(z-1)(z+1)}$$

So three singularities: $z_1 = 1, z_2 = -1, z_3 = 0$, only z_3 is inside γ.

$m = 3$, so

$$\begin{aligned}
\operatorname{Res}[f, 0] &= \lim_{z\to 0} \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^3 \cdot f(z)) \\
&= \lim_{z\to 0} \frac{1}{2}\left(\frac{1}{z^2-1}\right)'' \\
&= \frac{1}{2}\lim_{z\to 0}\left(\frac{-2z}{(z^2-1)^2}\right)' \\
&= \lim_{z\to 0} \frac{-(z^2-1)^2 - (z)2(z^2-1)\cdot 2z}{(z^2-1)^4} \\
&= -1
\end{aligned}$$

So

$$\oint_\gamma \frac{1}{z^5 - z^3}\,\mathrm{d}z = 2\pi i \cdot \operatorname{Res}[f, 0] = -2\pi i$$

Unit V

(ii) Evaluate

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz, \quad \gamma = \{z : |z| = 2\}$$

We see that

$$f(z) = \frac{1}{(z+5)(z^2-1)} = \frac{1}{(z+5)(z+1)(z-1)}$$

We have poles at $z = -1, z = +1$ inside γ, so we write

$$\oint_\gamma f(z)\,dz = 2\pi i\{\text{Res}[f,-1] + \text{Res}[f,1]\}$$

We compute

$$\text{Res}[f,1] = \lim_{z\to 1}(z-1)\cdot\frac{1}{(z+5)(z^2-1)} = \lim_{z\to 1}\frac{1}{(z+5)(z+1)} = \frac{1}{12}$$

and

$$\text{Res}[f,-1] = \lim_{z\to -1}(z+1)\cdot\frac{1}{(z+5)(z^2-1)} = \lim_{z\to -1}\frac{1}{(z+5)(z-1)} = -\frac{1}{8}$$

Thus

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz = 2\pi i\left(\frac{1}{12} - \frac{1}{8}\right) = -\frac{\pi i}{12}$$

Integrals of the Form $\int_0^{2\pi} F(cos\theta, sin\theta)d\theta$

Put $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ We can then write

$dz = ie^{i\theta}d\theta$, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$. Since $dz = ie^{i\theta}d\theta = iz\,d\theta$ and $z^{-1} = 1/z = e^{-i\theta}$, these three quantities are equivalent to

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}(z+z^{-1}), \quad \sin\theta = \frac{1}{2i}(z-z^{-1}).$$

We have

$$\oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz},$$

where C is the unit circle $|z| = 1$.

EXAMPLE

Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\,d\theta.$

Solution When we use the substitutions given in (4), the given trigonometric integral becomes the contour integral

Unit V

$$\oint_C \frac{1}{(2+\frac{1}{2}(z+z^{-1}))^2}\frac{dz}{iz} = \oint_C \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2}\frac{dz}{iz}.$$

Carrying out the algebraic simplification of the integrand then yields

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\,dz.$$

From the quadratic formula we can factor the polynomial z^2+4z+1 as $z^2+4z+1=(z-z_1)(z-z_2)$, where $z_1=-2-\sqrt{3}$ and $z_2=-2+\sqrt{3}$. Thus, the integrand can be written

$$\frac{z}{(z^2+4z+1)^2} = \frac{z}{(z-z_1)^2(z-z_2)^2}.$$

Because only z_2 is inside the unit circle C, we have

$$\oint_C \frac{z}{(z^2+4z+1)^2}\,dz = 2\pi i\,\mathrm{Res}(f(z), z_2).$$

$$\mathrm{Res}(f(z), z_2) = \lim_{z\to z_2}\frac{d}{dz}(z-z_2)^2 f(z) = \lim_{z\to z_2}\frac{d}{dz}\frac{z}{(z-z_1)^2}$$
$$= \lim_{z\to z_2}\frac{-z-z_1}{(z-z_1)^3} = \frac{1}{6\sqrt{3}}.$$

Hence, $$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)}\,dz = \frac{4}{i}\cdot 2\pi i \mathrm{Res}(f(z), z_1) = \frac{4}{i}\cdot 2\pi i\cdot\frac{1}{6\sqrt{3}}$$

and, finally, $$\int_0^{2\pi}\frac{1}{(2+\cos\theta)^2}\,d\theta = \frac{4\pi}{3\sqrt{3}}.$$

Integrals of the Form $\int_{-\infty}^{+\infty} f(x)dx$

Suppose $y=f(x)$ is a real

function that is defined and continuous on the interval $[0, \infty)$. In elementary calculus the improper integral $I_1=\int_0^\infty f(x)\,dx$ is defined as the limit

$$I_1 = \int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx. \qquad (1)$$

If the limit exists, the integral I_1 is said to be **convergent**; otherwise, it is **divergent**. The improper integral $I_2=\int_{-\infty}^0 f(x)\,dx$ is defined similarly:

$$I_2 = \int_{-\infty}^0 f(x)\,dx = \lim_{R\to\infty}\int_{-R}^0 f(x)\,dx. \qquad (2)$$

Finally, if f is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)\,dx$ is defined to be

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = I_1 + I_2, \qquad (3)$$

provided *both* integrals I_1 and I_2 are convergent. If either one, I_1 or I_2, is divergent, then $\int_{-\infty}^{\infty} f(x)\,dx$ is divergent. It is important to remember that the right-hand side of (3) is *not* the same as

Unit V

$$\lim_{R\to\infty}\left[\int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx\right] = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \qquad (4)$$

For the integral $\int_{-\infty}^{\infty} f(x)\,dx$ to be convergent, the limits (1) and (2) must exist independently of one another. *But*, in the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges, we can then evaluate it by means of the single limiting process given in (4):

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \qquad (5)$$

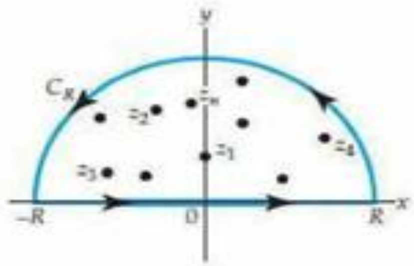
On the other hand, the symmetric limit in (5) may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is divergent. For example, the integral $\int_{-\infty}^{\infty} x\,dx$ is divergent since $\lim_{R\to\infty}\int_0^R x\,dx = \lim_{R\to\infty}\frac{1}{2}R^2 = \infty$. However, (5) gives

$$\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\frac{1}{2}[R^2 - (-R)^2] = 0. \qquad (6)$$

The limit in (5), if it exists, is called the **Cauchy principal value** (P.V.) of the integral and is written

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \qquad (7)$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)\,dx$, where the rational function $f(x) = p(x)/q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = p(z)/q(z)$ in the upper half-plane $\text{Im}(z) > 0$.



we have

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\sum_{k=1}^{n}\text{Res}(f(z), z_k),$$

where z_k, $k = 1, 2, \ldots, n$ denotes poles in the upper half-plane. If we c show that the integral $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$, then we have

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = 2\pi i\sum_{k=1}^{n}\text{Res}(f(z), z_k).$$

Unit V

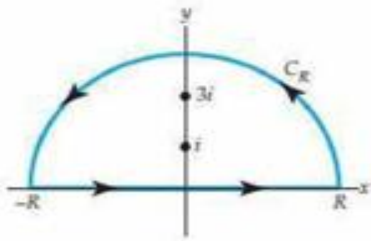
EXAMPLE

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$.

Solution Let $f(z) = 1/(z^2+1)(z^2+9)$. Since

$$(z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i),$$

we take C be the closed contour consisting of the interval $[-R, R]$ on the



x-axis and the semicircle C_R of radius $R > 3$.

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz$$

$$= I_1 + I_2$$

and $$I_1 + I_2 = 2\pi i\,[\text{Res}(f(z), i) + \text{Res}(f(z), 3i)].$$

At the simple poles $z = i$ and $z = 3i$ we find, respectively,

$$\text{Res}(f(z),\ i) = \frac{1}{16i} \quad \text{and} \quad \text{Res}(f(z),\ 3i) = -\frac{1}{48i},$$

so that $$I_1 + I_2 = 2\pi i\left[\frac{1}{16i} + \left(-\frac{1}{48i}\right)\right] = \frac{\pi}{12}.$$

This last result shows that $|I_2| \to 0$ as $R \to \infty$, and so we conclude that $\lim_{R\to\infty} I_2 = 0$.

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{12} \quad \text{or} \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{12}.$$

Cauchy's Residue Theorem:

Statement: Let D be a simply connected open subset of the complex plane, except for a finite number of isolated singularities.
If f(z) is analytic on D except at these singularities, and γ is a closed curve in D that does not pass through any singularity, then

Unit V

$$\oint_\gamma f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, z_k)$$

where z_k are the singularities inside γ, and $\mathrm{Res}(f, z_k)$ is the residue of f at z_k.

3. **Residue:**
   - The residue of f at $z = z_0$ is denoted by $\mathrm{Res}(f, z_0)$.
   - For a function $f(z)$ with a pole of order n at z_0, the residue is given by $\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}}[(z - z_0)^n f(z)]$
4. **Applications in Engineering:**
   - **Control Systems:** Residue theorem is used in the analysis of linear time-invariant systems and their transfer functions.
   - **Signal Processing:** In the analysis of signals and systems, especially in the Laplace transform domain.
   - **Electrical Engineering:** Used in the analysis of circuits with reactive components.
5. **Calculation of Residues:**
   - For a simple pole at z_0, the residue is given by $\mathrm{Res}(f, z_0) = \lim_{z \to z_0}(z - z_0)f(z)$.
   - For higher-order poles, derivatives are involved in the calculation.
6. **Special Cases:**
   - If γ is a simple, positively oriented closed contour, then

$$\oint_\gamma \frac{1}{z - z_0}\,dz = 2\pi i$$

where z_0 is inside γ.

- The theorem can also be applied to integrals over semi-circular contours or contours enclosing multiple singularities.

7. **Consequences:**
   - Simplifies the evaluation of certain definite integrals, especially those involving trigonometric functions and rational functions.