Unit IV

Limit of a complex function f(z):

Definition:

A function $w = f(z)$ tends to the **limit** ℓ as z tends to a point z_0 along **any path**, if to each positive arbitrary number ϵ, however small, there corresponds a positive number δ, such that

$|f(z) - \ell| < \epsilon$, whenever $0 < |z - z_0| < \delta$

i.e., $(\ell - \epsilon) < f(z) < (\ell + \epsilon)$, whenever $(z_0 - \delta) < z < (z_0 + \delta)$, $z \neq z_0$

and we write $\underset{z \to z_0}{\text{Lt}} f(z) = \ell$.

Continuity of f(z):

A single valued function $w = f(z)$ is said to be **continuous** at a point $z = z_0$, if $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

In other words:

A function f(z) is said to be continuous at a point z_0 if $f(z_0)$ exists, $\lim_{z \to z_0} f(z)$ exists and $\lim_{z \to z_0} f(z) = f(z_0)$.

i.e., limiting value of f(z) as z approaches z_0 coincides with the value $f(z_0)$.

A function f(z) is said to be continuous in any region R of the z-plane, if it is continuous at every point of the region.

This means that, a function is said to be continuous in a domain if it is continuous at every point of the domain.

A function, which is not continuous at z_0, is known as discontinuous at z_0.

This means that a function, in which $f(z_0)$ does not exist, or $\lim_{z \to z_0} f(z)$ does not exist or $\lim_{z \to z_0} f(z) \neq f(z_0)$, is known as discontinuous at z_0.

Result 1: If f(z) and g(z) are continuous function in D. Then their sum $f + g$, difference $f - g$, product fg, quotient f/g are all continuous in D. Continuous function of a continuous function is continuous.

Result 2: $f = u + iv$ is continuous if both u and v are continuous.

Derivative of f(z) (Differentiability) :

Let $w = f(z)$ be a single valued function of the variable $z (= x + iy)$, then the derivative or differential coefficient of $w = f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z + \delta z) - f(z)}{\delta z}$$

provided the limit exists and has the **same value** for all the different ways in which $\delta z \to 0$.

In other words:

A function f(z) is said to be differentiable at a point z_0 if the limit

$$f'(z_0) = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (\text{with } z = z_0 + \delta z)$$

exists.

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The limit $f'(z_0)$ is known as the derivative of f(z) at z_0.

The above limit should be the same along any path from z to z_0.

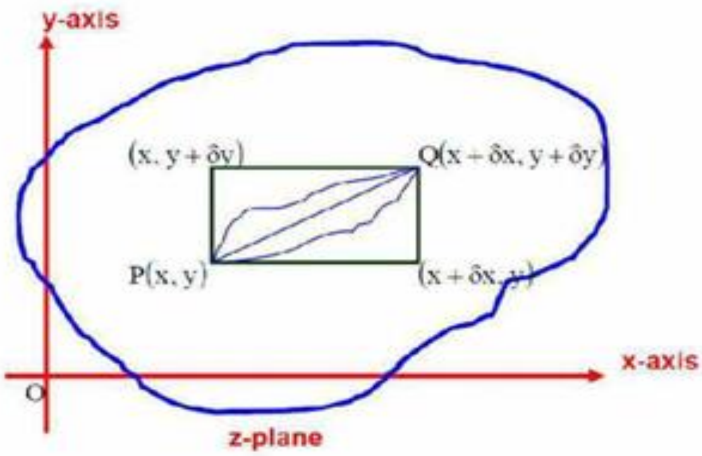
Thus, differentiability of a complex function is a stringent requirement.

Differentiation rules:

Differentiation rules of real calculus are valid in complex differentiation also.

1. $\frac{dc}{dz} = 0$ where c = complex constant
2. $\frac{d}{dz}[f \pm g] = \frac{df}{dz} \pm \frac{dg}{dz}$
3. $\frac{d}{dz}[cf(z)] = c\frac{df}{dz}$
4. $\frac{d}{dz}[f.g] = f\frac{dg}{dz} + \frac{df}{dz}.g$
5. $\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dg} - f\frac{dg}{dz}}{g^2}$
6. a. $\frac{d}{dz}[f(z)]^n = n[f(z)]^{n-1}\frac{df}{dz}$

   b. $\frac{d}{dz}z^n = nz^{n-1}$
7. Chain rule $\frac{dw}{dz} = \frac{dw}{d\zeta}.\frac{d\zeta}{dz}$ if $w = f(\zeta)$ and $\zeta = g(z)$.



Remarks: Suppose P(z) is fixed and $Q(z+\delta z)$ neighbouring point. The point Q may approach P along any straight or curved path in the given region, i.e., δz may tends to zero in any manner and $\frac{dw}{dz}$ may not exist. Then it becomes a fundamental problem to determine the necessary and sufficient conditions for $\frac{dw}{dz}$ to exist. This fact is settled by the following theorem.

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Q.No.1.: Show that $\lim_{z\to 0}\frac{x^2y}{x^4+y^2}$ does not exist even though this function approached the same limit along every straight line through the origin.

Sol.: Path I. $\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{x^4+y^2}=\lim_{y\to 0}\frac{0}{y^2}=\lim_{y\to 0}0=0$

Path II. $\lim_{\substack{y\to 0\\ x\to 0}}\frac{x^2y}{x^4+y^2}=\lim_{x\to 0}\frac{0}{x^4}=\lim_{x\to 0}0=0$

Path III. Along any straight line through origin.

Let $y=mx$

$$\lim_{\substack{y=mx\\ x\to 0}}\frac{x^2y}{x^4+y^2}=\lim_{x\to 0}=\frac{mx^3}{x^4+m^2x^2}=\lim_{x\to 0}\frac{mx}{x^2+m^2}=0$$

Path IV as $y=mx^2$, then

$$\lim_{\substack{y=mx^2\\ x\to 0}}\frac{x^2y}{x^4+y^2}=\lim_{x\to 0}=\frac{m.x^4}{x^4+m^2x^4}=\lim_{x\to 0}\frac{m}{1+m^2}=\frac{m}{1+m^2}\neq 0$$

and different for different values of m.

Therefore, the limit does not exist.

Q.No.2.: Determine where the given function is continuous

(a) $\frac{1}{1+z^2}$, (b). $\frac{1}{z-1}$ inside a unit circle.

How about in the complex plane.

Sol.: $\frac{1}{1+z^2}$ is continuous everywhere except where $1+z^2=0$, i.e., at $z=\pm i$.

When unit circle is considered, $|z|<1$, $z=\pm i$ are excluded.

Thus $\frac{1}{1+z^2}$ is continuously inside $|z|=1$.

Similarly, $\frac{1}{z-1}$ is also continuous inside $|z|=1$.

If the entire complex plane is considered, then both $\frac{1}{1+z^2}$ and $\frac{1}{z-1}$ are discontinuous, at $z=\pm i$ and $z=1$ respectively.

Analytic functions:

Definition:

If a single-valued function f(z) possesses a unique derivative w.r.t. z at all points of a region R, then f(z) is called an analytic function.

In other words:

A function f(z) is said to be analytic at a point z_0 if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0. An analytic function is also known as "holomorphic function", "regular function", "monogenic function".

A function f(z) is analytic in a domain if it is analytic every point of the domain.

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A point at which an analytic function ceases to possess a derivative is called a singular point of the function.

Thus, if u and v are real single-valued functions of x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous throughout a region R, then the Cauchy's-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$,

are both necessary and sufficient conditions for the function $f(z) = u + iv$ to be analytic in the region R.

The derivative of f(z) is then given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Thus Cauchy's-Riemann equations (or conditions) are used to determine whether a complex function is analytic or not.

Note:

1. If f is analytic in a domain D, then u, v satisfy C-R equations at all points in D.
2. C-R conditions are **necessary**, but not sufficient conditions.
3. C-R conditions are **sufficient** if the partial derivatives are continuous, i.e., u(x,y), v(x,y) have continuous first partial derivatives and satisfy C-R equations then f = u + iv is analytic.

f analytic → C-R conditions – Continuous P.D. → Analyticity

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Entire Function:

A function which is analytic everywhere, for all z in the complex plane, is known as entire function.

e.g., Polynomials, rational functions are entire functions, means analytic everywhere.

$|z|^2$ is differentiable only at z = 0. So, it is nowhere analytic.

Thus, analyticity is very stringent condition

Properties of analytic functions:

1. If f(z) and g(z) are analytic, then $f+g, fg, f/g$ are analytic if $g(z) \neq 0$.
2. Analytic function of an analytic function is analytic.
3. An entire function of an entire function is entire.
4. If f is analytic, then it is continuous (analyticity) $\Rightarrow$ differentiability $\Rightarrow$ continuity.
5. Derivative of an analytic function is itself analytic.

   Proof. $f' = u_x + iv_x = U + iV$.

   f is analytic, so $u_x = v_y, u_y = -v_x$

   Differentiating w.r.t. and y, we get

   $u_{xx} = v_{yx}, u_{yy} = -v_{xy}$ or $U_x = V_x$ and $U_y = -V_x$.

   i.e. U, V satisfy CR conditions. Hence f' is analytic.
6. If $f = u + iv$ is analytic, then the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are mutually orthogonal, i.e., $u - c_1$ are orthogonal trajectories of $v - c_2$ and vice-versa.

Proof: By implicit differentiation of $u = c_1$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial x} = 0 \Rightarrow \frac{\partial y}{\partial x} = -\frac{ux}{uy}.$$

Similarly $v_x + v_y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = \frac{-v_x}{v_y}$.

Product of slopes $= \frac{ux}{uy} \cdot \left(\frac{-v_x}{v_y} \right) = -1$ by CR conditions.

Remarks: The real and imaginary parts of an analytic function are called **conjugate functions**. Thus if, $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then u(x, y) and v(x, y) are conjugate functions. The relation between two conjugate functions is given by C-R equations.

Unit IV

[Necessary and Sufficient conditions for the derivative of the function f(z)]

Theorem:

The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R;

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

The conditions (ii) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

POLAR FORM OF CAUCHY-RIEMANN EQUATIONS

Show that the polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Also deduce that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

Proof: If (r, θ) be the co-ordinates of the point whose cartesian co-ordinates are (x, y), then

$z = x + iy = re^{i\theta}$.

$\therefore u + iv = f(z) = f(re^{i\theta})$, where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}).ire^{i\theta} = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{(i)} \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}. \quad \text{(ii)}$$

which is the required polar form of Cauchy-Riemann equations.

Now, differentiating (i) partially w. r. t. r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}. \quad \text{(iii)}$$

Differentiating (ii) partially w. r. t. θ, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r \partial \theta}. \quad \text{(iv)}$$

Thus using (i), (ii) and (iv), we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right) = 0. \left[\because \frac{\partial^2 v}{\partial \theta \partial r} = \frac{\partial^2 v}{\partial r \partial \theta}\right]$$

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Q-1

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} \quad (z\neq 0),\ f(0)=0,$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

Sol.: Here $f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\ (z\neq 0)$.

$\therefore \lim_{z\to 0} f(z)=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}=\lim_{y\to 0}\frac{-y^3(1-i)}{y^2}=\lim_{y\to 0}[-y(1-i)]=0.$

Also $\lim_{z\to 0} f(z)=\lim_{\substack{y\to 0\\ x\to 0}}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}=\lim_{x\to 0}\frac{x^3(1+i)}{x^2}=\lim_{x\to 0}[x(1+i)]=0.$

Also $f(0)=0$ (given)

Thus $\lim_{z\to 0} f(z)=f(0)$ when $x\to 0$ first and then $y\to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y=mx$. Then

$$\lim_{z\to 0} f(z)=\lim_{\substack{y\to mx\\ x\to 0}}\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}=\lim_{x\to 0}\frac{x^3(1+i)-m^3x^3(1-i)}{(1+m^2)x^2}$$

$$=\lim_{x\to 0}\frac{x[1+i-m^3(1-i)]}{1+m^2}=0.$$

Hence $\lim_{z\to 0} f(z)=f(0)$, in whatever manner $z\to 0$.

$\Rightarrow f(z)$ is continuous at the origin.

Hence f(z) is continuous for all values of z.

Now $f(z)=\frac{x^3-y^3}{x^2+y^2}+i\frac{x^3+y^3}{x^2+y^2}=u(x,y)+iv(x,y)$

Since $f(0)=0 \Rightarrow u(0,0)=0$, and $v(0,0)=0$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{0,0}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{x\to 0}\frac{-y}{y}=-1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0}=\lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x}=\lim_{x\to 0}\frac{x}{x}=1$$

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and $\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y\to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \underset{x\to 0}{\text{Lt}} \frac{y}{y} = 1$

$\rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Thus, the C-R equations are satisfied at the origin.

But $f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z\to 0}{\text{Lt}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$.

If $z \to 0$ along the path $y = mx$, then $f'(0) = \frac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$,

which assume different values as m varies. So $f'(z)$ is not unique at (0, 0).

Thus, f(z) is not analytic at the origin even though it is continuous and satisfies the C.R. equations at the origin.

Q-2 Show that

$$f(z) = \begin{cases} \frac{x^2y^3(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin, even though CR equations are satisfied at the origin.

Sol.: Let $z \neq 0$ and given $f(z) = \frac{x^3y^3}{x^6+y^{10}} + i\frac{x^2y^4}{x^6+y^{10}} = u + iv$ (say).

Here $u = \frac{x^3y^3}{x^6+y^{10}}$, $v = \frac{x^2y^4}{x^6+y^{10}}$.

Now $\left(\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{x\to 0}{\text{Lt}} \frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{0-0}{x} = 0$.

$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{y\to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{\text{Lt}} \frac{0-0}{y} = 0$.

$\left(\frac{\partial v}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{y\to 0}{\text{Lt}} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{\text{Lt}} \frac{0-0}{y} = 0$.

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$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{x\to 0}{\text{Lt}} \frac{v(x,0)-(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{0-0}{x} = 0.$$

$\Rightarrow u_x = v_y,\ u_y = -v_x.$

Hence, CR equations are satisfied at the origin.

Further $f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{\frac{x^3y^3}{x^6+y^{10}} + i\frac{x^2y^4}{x^6+y^{10}} - 0}{z}.$

Choose the path $y = x$, then

$$f'(0) = \underset{x\to 0}{\text{Lt}}\left(\frac{x^6}{x^6+x^{10}} + i\frac{x^6}{x^6+x^{10}}\right)\frac{1}{x+ix}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{(1+i)x^6}{x^6(1+x^4)(1+i)x} = \underset{x\to 0}{\text{Lt}} \frac{1}{x(1+x^4)} = \frac{1}{0} = \infty.$$

$\Rightarrow f'(0)$ does **not exist** at origin. $\Rightarrow f(z)$ is not analytic at the origin.

Unit IV

Harmonic Functions:

Definition: A function f(x, y), is said to be a harmonic function if it satisfies the Laplace's equation, i.e., $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, i.e., $\nabla^2 f=0$.

If $f(z)=u+iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i. e. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ (i) and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ (ii)

Differentiate (i) w. r. t. x and (ii) w. r. t. y, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x \partial y} \quad \text{(iii)}$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y \partial x} \quad \text{(iv)}$$

Adding (iii) and (iv) and assuming that $\frac{\partial^2 v}{\partial x \partial y}=\frac{\partial^2 v}{\partial y \partial x}$, we get $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (v)

Similarly, by differentiating (i) w. r. t. y and (ii) w. r. t. x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \quad \text{(vi)}$$

Thus, both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as harmonic functions and their theory is called potential theory.

Thus, a function f(x, y), is said to be a harmonic function if it satisfies the Laplace's equation, i.e., $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, i.e., $\nabla^2 f=0$.

Orthogonal system: Consider the two families of curves

$$u(x, y)=c_1 \quad \text{(i)}$$

$$v(x, y)=c_2 \quad \text{(ii)}$$

Differentiating (i), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{\partial v}{\partial y}}{\frac{\partial v}{\partial x}}=m_1 \text{ (say)} \quad \text{[by C-R equations]}$$

Similarly (ii) gives $\frac{dy}{dx}=-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}=m_2$ (say)

$\therefore m_1 m_2=-1$, i. e. (i) and (ii) form an orthogonal system.

Hence every function $f(z)=u+iv$ defines two families of curves $u(x, y)=c_1$ and $v(x, y)=c_2$, which form an orthogonal system.

Unit IV

Q-1

Show that an analytic function with constant modulus is constant.

or

Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Sol.: Let $f(z) = u + iv$.

Since $|f(z)| = \text{constant} = c$, say, then we have

$\sqrt{u^2+v^2} = c \Rightarrow u^2+v^2 = c^2$

$\therefore 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$ and $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$

$\Rightarrow u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$ and $u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$. [by C-R equations]

Eliminating $\frac{\partial u}{\partial y}$, we get $(u^2+v^2)\frac{\partial u}{\partial x} = 0$.

Thus provided $w = u + iv \neq 0$.

Similarly, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$.

Since, the four partial derivatives of u and v are zero.

$\Longrightarrow$ The functions u, v are constant.

$\Longrightarrow$ $w = u + iv$ is also constant.

This completes the proof.

Q-2

Is the function $u(x, y) = 2xy + 3xy^2 - 2y^2$ harmonic (i.e., solution of Laplace equation)?

Sol.: $u_x = 2y + 3y^2$, $u_{xx} = 0$, $u_y = 2x + 6xy - 6y^2$, $u_{yy} = 6x - 12y$.

So $u_{xx} + u_{yy} \neq 0$. Therefore u is not harmonic.

Conformal Mapping (Transformation):

Suppose two curves C, C_1 in the z-plane intersect at the point P and the corresponding curves C' and C_1' in the w-plane intersect at P'. If the angle of intersection of the curves at P is the **same** as the angle of intersection of the curves at P' in magnitude and sense, then the **transformation** is said to be **conformal** at P.

A transformation, which preserves angles both in magnitude and sense between every pair of curves through a point, is said to conformal at the point. But basically, conformal mapping has two kinds:

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1. Conformal mapping of the first kind:

If the conformal mapping preserves angles both in magnitude and sense, then conformal mapping is known as conformal mapping of the first kind.

2. Conformal mapping of the second kind (Isogonal mapping):

If the conformal mapping preserve angles only in magnitude but not in sense, which is reserved, like $w - \bar{z}$, where $\arg \bar{z} = -\arg z$, then conformal mapping is known as conformal mapping of the second kind.

Remarks: Given two mutually orthogonal one-parameter families of curves say $\phi(x,y) = c_1$ and $\psi(x,y) = c_2$, then their image curves in the w-plane $\phi(u,v) = c_3$ and $\psi(u,v) = c_4$ under a conformal mapping are also mutually orthogonal. Thus, conformal mapping preserves the property of mutual orthogonality of system of curves in the plane.

Note: Conformal mapping is used to map complicated regions conformally onto simpler, standard regions such as circular disks, half planes and strips for which the boundary value problems are easier.

Condition under which the transformation $w = f(z)$ is conformal

Condition for Conformality:

Theorem:

A mapping $w=f(z)$ is conformal at each point z_0 where $f(z)$ is analytic and $f'(z_0) \neq 0$.

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Remarks:

(1). Relation (iv), i.e. $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an angle $\phi = \text{amp}\{f'(z)\}$ under the given transformation.

(2). The relation (iii), i.e. $\rho = \underset{\delta z \to 0}{\text{Lt}} \frac{r'}{r}$ shows that in the conformal transformation, the lengths of arcs passing through P any direction are magnified in the ratio $\rho : 1$, where $\rho = |f'(z)|$. Thus, an infinitesimal length in the z-plane is magnified by the factor $|f'(z)|$ in the w-plane and consequently the infinitesimal areas are in the z-plane magnified by the factor $|f'(z)|^2$ in the w-plane in a conformal transformation.

(3). Jacobian of a transformation:

If $w = f(z)$, i.e., $u + iv = f(x + iy)$ is an analytic function which maps a closed region D of the z-plane into a closed region D' of w-plane, then u and v must satisfy C-R equation.

$$\therefore J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{-\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right| = |f'(z)|^2.$$

Hence, in a conformal transformation, infinitesimal areas are magnified by the factor $J\left(\frac{u, v}{x, y}\right)$. Also the condition of a conformal mapping is $J\left(\frac{u, v}{x, y}\right) \neq 0$.

Unit IV

Conformal Mapping by Elementary Functions:

General Linear Transformation or Linear Transformation

General Linear Transformation or simply linear transformation defined by the function

$w = f(z) = az + b$ (i)

($a \neq 0$, and b are arbitrary complex constants) maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic and $f'(z) = a \neq 0$ for any z. If a = 0, (i) reduces to a constant function.

Special cases of linear transformation are:

i. Identity Transformation:

$$w = z$$

for $a = 1, b = 0$, which maps a point z onto itself.

ii. Translation:

$$w = z + b$$

for a = 1, which translates (shifts) z through a distance $|b|$ in the direction b.

If $z = x + iy$, $b = b_1 + ib_2$ and $w = u + iv$,

then the transformation becomes $u + iv = x + iy + b_1 + ib_2 \Rightarrow u = x + b_1$ and $v = y + b_2$.

i.e., the point P(x, y) in the z-plane is mapped onto the point $P'(x + b_1, y + b_2)$ in the w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus, if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector b.

Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size. Thus, this transformation is a mere translation of the axis and preserves the shape and size. In particular, this transformation changes circles into circles.

iii. Magnification and Rotation:

$$w = e^{i\theta}.z$$

for $a = e^{i\theta}, b = 0$ which rotates (the radius vector of point) z through a scalar angle θ (counter clockwise if $\theta > 0$ while clockwise if $\theta < 0$)

Unit IV

Let $w = cz$, where c is complex constant.

Let $c = \rho e^{i\alpha}$, $z = re^{i\theta}$ and $w = Re^{i\phi}$, then the transformation becomes

$Re^{i\phi} = \rho e^{i\alpha}.re^{i\theta} = \rho re^{i(\theta+\alpha)} \rightarrow R = \rho r$ and $\phi = \theta + \alpha$.

Thus, the transformation maps a point $P(r, \theta)$ in the z-plane onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence this transformation consists of magnification (or contraction) of the radius vector P by $\rho = |c|$ and its rotation through an angle α = amp (c).

Accordingly, it maps any figure in the z-plane into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

iv. Stretching (scaling):

$$w = az$$

for 'a' real stretches if $a > 1$ (contracts if $0 < a < 1$) the radius vector by factor 'a'.

Thus, the linear transformation $w = f(z) = az + b$ consists of rotation through angle arg a, scaling by factor $|a|$, followed by translation through vector b. This transformation is used for constructing conformal mapping of 'similar' figures.

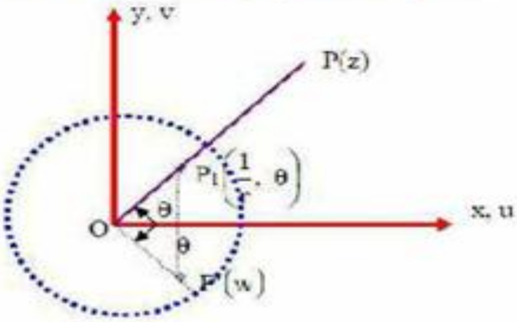
Result: Show that circles are invariant under translation, rotation and stretching.

i.e., Linear transformation preserves circles i.e., a circle in the z-plane under transformation maps to a circle in the w-plane.

Some standard transformations:

Inversion and Reflection: $w = \frac{1}{z}$.

Here, it is convenient to think the w-plane as superposed on z-plane.



Let $z = re^{i\theta}$ and $w = Re^{i\phi}$, then the transformation $w = \frac{1}{z}$ becomes $Re^{i\phi} = \frac{1}{r}e^{-i\theta}$

$\Rightarrow R = \frac{1}{r}$ and $\phi = -\theta$.

Thus, under the transformation $w = \frac{1}{z}$, a point $P(r, \theta)$ in z-plane is mapped into the point $P_1\left(\frac{1}{r}, -\theta\right)$.

Unit IV

Consider the w-plane superposed on the z-plane. If P is (r, θ) and $P_1 = \left(\frac{1}{r}, -\theta\right)$, then

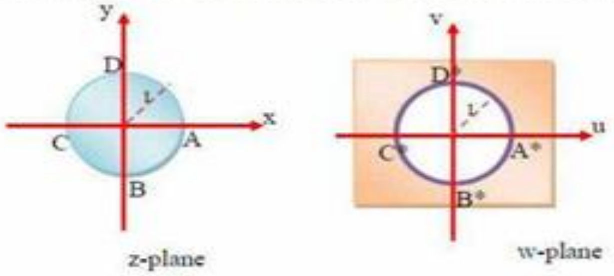
$$OP_1 = \frac{1}{r} = \frac{1}{OP} \Rightarrow OP.OP_1 = 1$$

$\Rightarrow P_1$ is the inverse of P w.r.t. the unit circle with centre O.

Then the reflection P' of P_1 in the real axis represents $w = \frac{1}{z}$.

Hence, this transformation is an inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Clearly, the function $w = \frac{1}{z}$ maps the interior of the unit circle $|z| = 1$ onto the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ onto the interior of $|w| = 1$.



However, the origin $z = 0$ is mapped to the point $w = \infty$, called the point at infinity.

Result: Prove that circles are invariant under $w = \frac{1}{z}$.

or

Show that the transformation $w = \frac{1}{z}$ always maps circles into circles.

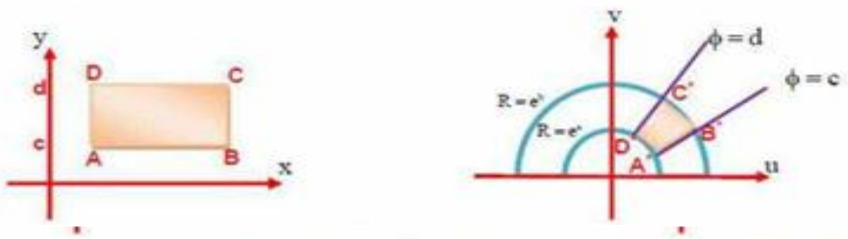
Transformation $w = e^z$:

Rewriting

$$Re^{i\phi} = w = e^z = e^{x+iy} = e^x.e^{iy}$$

Therefore $R = e^x$ and $\phi = y$ (i)

i.e., modulus of w is e^x and argument of w is y. The line $x = c$ = constant maps onto the circle $R = e^c$.



The line $y = c$ maps on to the ray $\phi = c$. Thus, the region $a \le x \le b$, $c \le y \le d$ in the z-plane is mapped to the region $A^*B^*C^*D^*$ in the w-plane bounded by the concentric circles $R = e^b$ and $R = e^a$ and by the rays $\phi = c$ and $\phi = d$.

Unit IV

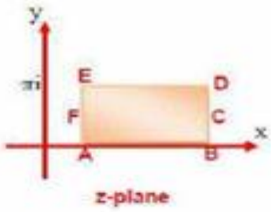
Note 1: Since $e^z \neq 0$, $w = 0$ is not mapped . Thus, the origin in w-plane is excluded.

Note 2: This mapping is one to one if $d - c < 2\pi$.

Particular case: $c = 0,\ d = \pi$.

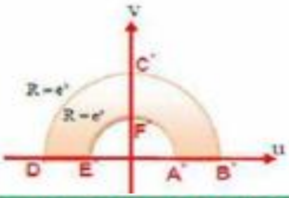
Consider the rectangular region in z-plane

$a \leq x \leq b,\ \ 0 \leq y \leq \pi$ (see figure)



By (i), $e^a \leq R = e^x \leq e^b$ and $0 \leq \phi = y \leq \pi$.

Thus, the rectangular region maps onto the upper half of the annulus ring $e^a < R < e^b$, $0 \leq \phi \leq \pi$. (see figure)



Bilinear Transformation or Mobius Transformation:

The transformation $w = \dfrac{az+b}{cz+d}$, (i)

where a, b, c and d are complex constants and $ad - bc \neq 0$ is called as **Bilinear transformation or Mobius transformation or linear fractional transformation.**

In other words, Bilinear transformation is the function w of a complex variable z of the form $w = f(z) = \dfrac{az+b}{cz+d}$.

Bilinear transformation is conformal for all z:

Differentiating (i) w.r.t., z, we get

$$\frac{dw}{dz} = \frac{ad - bc}{(cz+d)^2}$$

If $ad - bc \neq 0$, then $\dfrac{dw}{dz} \neq 0$ for any z and therefore Bilinear transformation is conformal for all z, i.e., it maps z-plane conformally onto the w-plane.

Thus, the condition $ad - bc \neq 0$ ensures that $\dfrac{dw}{dz} \neq 0$, i.e., the transformation is conformal.

If $ad - bc = 0$, then $\dfrac{dw}{dz} = 0$ for any z. Then every point of the z-plane is critical point and the function is not conformal.

Special cases of Bilinear transformation:

For a choice of constants a, b, c, d, we get special cases of Bilinear transformation as

$w = z + b$ Translation

$w = az$ Rotation

$w = az + b$ Linear transformation

$w = \dfrac{1}{z}$ inversion in the unit circle.

Thus B.T. can be considered as combination of these transformations.

Unit IV

Inverse of bilinear transformation $w = \dfrac{az+b}{cz+d}$:

Solving (i) for z, we find that inverse of the Bilinear transformation is

$$z = \frac{-dw+b}{cw-a} \qquad \text{(ii)}$$

which is also a bilinear transformation.

Remarks:

(1). From (i), we see that each point in the z-plane except $z = -\dfrac{d}{c}$, maps into a unique point in the w-plane. Similarly, (ii) shows that each point in the w-plane except $w = \dfrac{a}{c}$, maps into a unique point in the z-plane. Considering the two exceptional points as points at infinity in the two planes, we can say that there is one to one correspondence between all points in the two planes.

From (i), observe that the point $z = -\dfrac{d}{c}$ corresponds to $w = \infty$, point at infinity in the w-plane. Similarly from (ii), the point $w = \dfrac{a}{c}$ corresponds to $z = \infty$, point at infinity in the z-plane.

(2). Invariant points of bilinear transformation:

If z maps into itself in the w-plane

(i.e. $w = z$), then $z = \dfrac{az+b}{cz+d} \Rightarrow z = \dfrac{az+b}{cw+d} \Rightarrow cz^2 + (d-a)z - b = 0$.

The roots of this equation (say: z_1, z_2) are defined as the invariant or fixed points of the bilinear transformation $z = \dfrac{az+b}{cz+d}$.

In other words, fixed (or invariant) points of function $w = f(z)$ are points which are mapped onto themselves i.e., $w = f(z) = z$.

Example:

$w = z$ has every point a fixed point

$w = \bar{z}$ infinitely many

$w = \dfrac{1}{z}$ has two

$w = z + b$ has no fixed point

To obtain the fixed points of $w = \dfrac{az+b}{cz+d}$, solve

$z = \dfrac{az+b}{cz+d}$, which is a quadratic in z given by

$$cz^2 - (a-d)z - b - 0. \qquad \text{(i)}$$

Thus, the roots, say α, β of (i) are fixed points of $w = \dfrac{az+b}{cz+d}$.

If two roots of (i) are equal then bilinear transformation is said to be parabolic.

The quadratic with α, β as roots is $z^2-(\alpha+\beta)z+\alpha\beta=0$.

For any complex constant γ,

$$z^2-(\alpha+\beta)z+\gamma z-\gamma z+\alpha\beta=0$$

$$z(z-(\alpha+\beta-\gamma))-\gamma z-\alpha\beta$$

$$z=\frac{\gamma z-\alpha\beta}{z-(\alpha-\beta)+\gamma}.$$

Thus, the bilinear transformations, whose fixed points α, β are given by

$$w=\frac{\gamma z-\alpha\beta}{z-(\alpha+\beta)+\gamma}. \qquad \text{(ii)}$$

For various values γ, (ii) gives B.T. with fixed points α, β.

(3). In the linear transformation $w=\frac{az+b}{cz+d}$, $ad-bc\neq 0$, dividing the numerator and denominator of the right side by one of the four constants, we observe that there are only three independent constants. Hence, three conditions are required to determine a bilinear transformation. For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Cross-ratio or anharmonic ratio of four numbers:

The cross-ratio or anharmonic ratio of four numbers z_1, z_2, z_3, z_4 is the linear fraction given by

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Theorem: A bilinear transformation preserves cross-ratio of four points.

or

The cross ratio is invariant under a bilinear transformation.

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation $w=\frac{az+b}{cz+d}$.

This means that w_1, w_2, w_3, w_4 are respectively the images of z_1, z_2, z_3, z_4 under the bilinear transformation $w=\frac{az+b}{cz+d}$.

If these points are finite, then from $w=\frac{az+b}{cz+d}$, we have

$$w_j-w_k=\frac{az_j+b}{cz_j+d}-\frac{az_k+b}{cz_k+d}=\frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j-z_k)$$

Using this relation for j, k = 1, 2, 3, 4, we get

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)}=\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation.

Unit IV

Determination of Bilinear Transformation:

A bilinear transformation can be uniquely determined by three given conditions. Although four constants a, b, c, d appear in (i), essentially they are three ratio of three of these constants to the fourth one.

To find the unique bilinear transformation which maps three given points z_1, z_2, z_3 onto three distinct images w_1, w_2, w_3, consider w which is the image of a general point z under this transformation. Now as we know the cross-ratio is preserved under bilinear transformation, so the cross-ratio of the four points w_1, w_2, w_3, w must be equal to the cross-ratio of z_1, z_2, z_3, z.

Hence, the unique bilinear transformation that maps three given points z_1, z_2, z_3 onto three given images w_1, w_2, w_3 is given by

$$\frac{(w_1-w_2)(w_3-w)}{(w_1-w)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z)}{(z_1-z)(z_3-z_2)}$$

Note 1: If one of the points, say: $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 i.e., $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(z_3-z_4)}{(z_3-z_2)}$.

Milne's Thompson Method:

Introduction:

- Milne's Thompson method is used to find analytic functions by solving partial differential equations.

Basic Idea:

- Given a partial differential equation involving a function f(z), Milne's Thompson method transforms the equation into a simpler form by introducing new variables.

Procedure:

- Choose a transformation that simplifies the differential equation. This often involves introducing new variables or using a change of coordinates.
- Apply the chosen transformation to the given partial differential equation.
- Solve the transformed equation to find the analytic function.

4. **Example:**

- Consider a partial differential equation involving $f(z)$: $\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$.
- Apply Milne's Thompson method to transform the equation and make it easier to solve.

Unit IV

Ex. solve it using Milne's Thompson Method

$$4\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 3y = 0$$

1. **Characteristic Equation:**

   The characteristic equation for this ODE is obtained by substituting $y = e^{mx}$ into the ODE:

   $4m^2 + 8m + 3 = 0$

   Solve this quadratic equation to find the roots m_1 and m_2.

2. **General Solution:**

   The general solution is given by:

   $y(x) = C_1e^{m_1x} + C_2e^{m_2x}$

   where C_1 and C_2 are constants determined by initial or boundary conditions.

Now, let's assume that the roots of the characteristic equation are $m_1 = -1$ and $m_2 = -3$.

1. **Specific Solution:**

   The specific solution is obtained by substituting the given roots into the general solution:

   $y(x) = C_1e^{-x} + C_2e^{-3x}$

   To determine the values of C_1 and C_2, you would need additional information such as initial conditions or boundary conditions.

   For instance, if $y(0) = 2$ and $y'(0) = -1$, you can substitute these values into the specific solution and solve for C_1 and C_2.

   $y(0) = C_1 + C_2 = 2$

   $y'(0) = -C_1 - 3C_2 = -1$

   Solve this system of equations to find C_1 and C_2.