Unit V: DIFFERENTIAL CALCULUS-II

Successive differentiation :-

INTRODUCTION

Calculus is one of the most beautiful intellectual achievements of human being. The mathematical study of change motion, growth or decay is calculus. One of the most important idea of differential calculus is derivative which measures the rate of change of a given function. Concept of derivative is very useful in engineering, science, economics, medicine and computer science.

The first order derivative of y denoted by $\frac{dy}{dx}$, second order derivative, denoted by $\frac{d^2y}{dx^2}$ third order derivative by $\frac{d^3y}{dx^3}$ and so on. Thus by differentiating a function $y = f(x)$, n times, successively, we get the nth order derivative of y denoted by $\frac{d^ny}{dx^n}$ or D^ny or $y_n(x)$. Thus, the process of finding the differential co-efficient of a function again and again is called **Successive Differentiation.**

*n*th DERIVATIVE OF SOME ELEMENTARY FUNCTIONS

1. Power Function $(ax + b)^m$

Let

$$y = (ax+b)^m$$
$$y_1 = ma(ax+b)^{m-1}$$
$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$

....

....

$$y_n = m(m-1)(m-2)\ldots(m-\overline{n-1})\,a^n(ax+b)^{m-n}$$

Case I. When m is positive integer, then

$$y_n = \frac{m(m-1)\ldots(m-n+1)(m-n)\ldots3\cdot2\cdot1}{(m-n)\ldots3\cdot2\cdot1}\,a^n(ax+b)^{m-n}$$

$$\Rightarrow \quad \boxed{y_n = \frac{d^n}{dx^n}(ax+b)^m = \frac{\lfloor m}{\lfloor m-n}\,a^n(ax+b)^{m-n}}$$

Case II. When $m = n =$ +ve integer

$$y_n = \frac{\lfloor n}{\lfloor 0}\,a^n(ax+b)^0 = \lfloor n\, a^n \Rightarrow \boxed{\frac{d^n}{dx^n}(ax+b)^n = \lfloor n\, a^n}$$

Unit V: DIFFERENTIAL CALCULUS-II

Case III. When $m = -1$, then

$$y = (ax+b)^{-1} = \frac{1}{(ax+b)}$$

$$\therefore \quad y_n = (-1)(-2)(-3)\ldots(-n)\, a^n (ax+b)^{-1-n}$$

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\left\{\frac{1}{ax+b}\right\} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}}$$

Case IV. Logarithm case: When $y = \log(ax+b)$, then

$$y_1 = \frac{a}{ax+b}$$

Differentiating $(n-1)$ times, we get

$$y_n = a^n \frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1}$$

Using case III, we obtain

$$\Rightarrow \quad \boxed{\frac{d^n}{dx^n}\{\log(ax+b)\} = \frac{(-1)^{n-1} \lfloor (n-1)\, a^n}{(ax+b)^n}}$$

2. Exponential Function

(*i*) Consider $y = a^{mx}$

$$y_1 = ma^{mx} . \log_e a$$

$$y_2 = m^2 a^{mx} (\log_e a)^2$$

....

....

$$\boxed{y_n = m^n a^{mx} (\log_e a)^n}$$

(*ii*) Consider $y = e^{mx}$

Putting $a = e$ in above $\boxed{y_n = m^n e^{mx}}$

3. Trigonometric Functions cos $(ax+b)$ or sin $(ax+b)$

Let $y = \cos(ax+b)$, then

$$y_1 = -a \sin(ax+b) = a \cos\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2 = -a^2 \cos(ax+b) = a^2 \cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = +a^3 \sin(ax+b) = a^3 \cos\left(ax+b+\frac{3\pi}{2}\right)$$

....

....

Unit V: DIFFERENTIAL CALCULUS-II

$$\boxed{y_n = \frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)}$$

Similarly, $$\boxed{y_n = \frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)}$$

4. Product Functions $e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$

Consider the function $y = e^{ax}\sin(bx+c)$

$$y_1 = e^{ax}.b\cos(bx+c) + ae^{ax}\sin(bx+c)$$
$$= e^{ax}[b\cos(bx+c) + a\sin(bx+c)]$$

To rewrite this in the form of sin, put

$$a = r\cos\phi,\ b = r\sin\phi, \text{ we get}$$
$$y_1 = e^{ax}[r\sin\phi\cos(bx+c) + r\cos\phi\sin(bx+c)]$$
$$y_1 = re^{ax}\sin(bx+c+\phi)$$

Here, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$

Differentiating again w.r.t. x, we get

$$y_2 = rae^{ax}\sin(bx+c+\phi) + rbe^{ax}\cos(bx+c+\phi)$$

Substituting for a and b, we get

$$y_2 = re^{ax}.\ r\cos\phi\sin(bx+c+\phi) + re^{ax}\ r\sin\phi\cos(bx+c+\phi)$$
$$y_2 = r^2e^{ax}[\cos\phi\sin(bx+c+\phi) + \sin\phi\cos(bx+c+\phi)]$$
$$= r^2e^{ax}\sin(bx+c+\phi+\phi)$$

$\therefore$ $$y_2 = r^2e^{ax}\sin(bx+c+2\phi)$$

Similarly, $$y_3 = r^3e^{ax}\sin(bx+c+3\phi)$$

....

$$\boxed{y_n = \frac{d^n}{dx^n}e^{ax}\sin(bx+c) = r^n e^{ax}\sin(bx+c+n\phi)}$$

In similar way, we obtain

$$\boxed{y_n = \frac{d^n}{dx^n}e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\phi)}$$

Example 1. Find the nth derivative of $\dfrac{1}{1-5x+6x^2}$

Sol. Let $$y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$$

$$y = \frac{2}{2x-1} - \frac{3}{3x-1} \qquad \text{(By Partial fraction)}$$

$\therefore$ $$y_n = 2\frac{d^n}{dx^n}(2x-1)^{-1} - 3\frac{d^n}{dx^n}(3x-1)^{-1}$$

$$= 2\left[\frac{(-1)^n \lfloor n \, 2^n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n \lfloor n \, 3^n}{(3x-1)^{n+1}}\right] \quad \left| \text{As} \frac{d^n}{dx^n}(ax+b)^{-1} = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}} \right.$$

or $$y_n = (-1)^n \lfloor n \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right].$$

Example Find the nth derivative of $e^{ax}\cos^2 x \sin x$.

Sol. Let $$y = e^{ax}\cos^2 x \sin x = e^{ax}\frac{(1+\cos 2x)}{2}\sin x$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{2\times 2}e^{ax}(2\cos 2x \sin x)$$

$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}\{\sin(3x) - \sin x\}$$

or $$y = \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$$

$\therefore$ $$y_n = \frac{1}{4}\left[r^n e^{ax}\sin(x+n\phi)\right] + \frac{1}{4}\left[r_1{}^n e^{ax}\sin(3x+n\theta)\right].$$

where $r = \sqrt{a^2+1}$; $\tan\phi = 1/a$

and $r_1 = \sqrt{a^2+9}$; $\tan\theta = 3/a$.

Find the nth differential co-efficient of $e^x . \sin^3 x$.

Solution. We have, $y = e^x \sin^3 x$.

We know that,

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$\Rightarrow \quad \sin^3 x = \frac{1}{4}[3\sin x - \sin 3x]$$

Let $$y = e^x \sin^3 x$$

$$= e^x \frac{1}{4}[3\sin x - \sin 3x] = \frac{3}{4}e^x \sin x - \frac{1}{4}.e^x \sin 3x$$

Differentiating n times, we get

$$y_n = \frac{3}{4}(1^2+1^2)^{\frac{n}{2}}.e^x \sin\left(x + n\tan^{-1}\frac{1}{1}\right) - \frac{1}{4}(1^2+3^2)^{\frac{n}{2}}.e^x \sin\left[3x + n\tan^{-1}\frac{3}{1}\right]$$

$$= \frac{3}{4}2^{\frac{n}{2}}e^x \sin\left(x+\frac{n\pi}{4}\right) - \frac{1}{4}.10^{\frac{n}{2}}.e^x \sin(3x + n\tan^{-1}3)$$ **Ans.**

Find the 30th derivative of $(2x+3)^{-1}$

Solution. Let $y = (2x+3)^{-1}$

$$\Rightarrow \quad y_{30} = (-1)^{30} (2x+3)^{-1-30} (2)^{30} (30!)$$

$$= \frac{2^{30}}{(2x+3)^{31}} (30!)$$

LEIBNITZ'S* THEOREM

Statement. If u and v be any two functions of x, then

$$D^n (u.v) = {}^nc_0 D^n (u).v + {}^nc_1 D^{n-1}(u).D(v) + {}^nc_2 D^{n-2}(u).D^2(v) + \ldots$$
$$+ {}^nc_r D^{n-r}(u).D^r(v) + \ldots + {}^nc_n u.D^n v .$$

Find the nth derivative of $x^{n-1} \log x$.

Solution. Here, we have

$$y = x^{n-1} \log x \qquad \ldots(1)$$

Differentiating (1) w.r.t. 'x', we get

$$y_1 = (n-1) x^{n-2} \log x + x^{n-1} \left(\frac{1}{x}\right)$$

$$x y_1 = (n-1) x^{n-1} \log x + x^{n-1}$$

$$x y_1 = (n-1) y + x^{n-1} \qquad \ldots(2)$$

Differentiating (2) $(n-1)$ times by Leibnitz's theorem, we get

$$x y_n + (n-1) y_{n-1} = (n-1) y_{n-1} + (n-1)! \qquad \ldots(3)$$

$$\Rightarrow \quad x y_n = (n-1)!$$

$$\Rightarrow \quad y_n = \frac{(n-1)!}{x}$$

Hence $$D^n \{x^{n-1} \log x\} = \frac{(n-1)!}{x}$$ **Ans.**

Unit V: DIFFERENTIAL CALCULUS-II

Simple Applications of Differentiation :-

Question 1: For the given curve: $y = 5x – 2x^3$, when x increases at the rate of 2 units/sec, then how fast is the slope of curve changes when x = 3?

Solution: Given that, $y = 5x – 2x^3$

Then, the slope of the curve, $dy/dx = 5-6x^2$

$\Rightarrow d/dt\ [dy/dx]= -12x.\ dx/dt$

$= -12(3)(2)$

= -72 units per second

Hence, the slope of the curve is decreasing at the rate of 72 units per second when x is increasing at the rate of 2 units per second.

Question 2: Show that the function $f(x) = \tan x – 4x$ is strictly decreasing on $[-\pi/3, \pi/3]$

Solution: Given that, $f(x) = \tan x – 4x$

Then, the differentiation of the function is given by:

$f'(x)= \sec^2 x – 4$

When $-\pi/3 < x\ \pi/3$, $1<\sec x <2$

Then, $1<\sec^2 x <4$

Hence, it becomes $-3 < (\sec^2 x-4)<0$

Hence, for $-\pi/3 < x\ \pi/3$, $f'(x)<0$

Therefore, the function "f" is strictly decreasing on $[-\pi/3, \pi/3]$

Unit V: DIFFERENTIAL CALCULUS-II

Question 3: A stone is dropped into a quiet lake and waves move in the form of circles at a speed of 4 cm/sec. At the instant, when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing?

Solution: We know that the area of a circle with radius "r" is given by $A = \pi r^2$.

Hence, the rate of change of area "A' with respect to the time "t" is given by:

$dA/dt = (d/dt)\, \pi r^2$

By using the chain rule, we get:

$(d/dr)(\pi r^2).\ (dr/dt) = 2\pi r.(dr/dt)$

It is given that, $dr/dt = 4$ cm/sec

Therefore, when $r = 10$ cm,

$dA/dt = 2\pi.\ (10).\ (4)$

$dA.dt = 80\ \pi$

Hence, when r = 10 cm, the enclosing area is increasing at a rate of $80\pi\ cm^2/sec$.

Question 4: What is the equation of the normal to the curve $y = \sin x$ at (0, 0)?

(a)x =0 (b) y=0 (c)x+y =0 (d)x-y=0

Solution: A correct answer is an option (c)

Explanation: Given that, $y = \sin x$

Hence, $dy/dx = \cos x$

Thus, the slope of the normal $= (-1/\cos x)_{x=0} = -1$

Therefore, the equation of the normal is $y-0 = -1(x-0)$ or $x+y=0$

Hence, the correct solution is option c.

Unit V: DIFFERENTIAL CALCULUS-II

Question 5: Determine all the points of local maxima and local minima of the following function: $f(x) = (-\frac{3}{4})x^4 - 8x^3 - (45/2)x^2 + 105$

Solution: Given function: $f(x) = (-\frac{3}{4})x^4 - 8x^3 - (45/2)x^2 + 105$

Thus, differentiate the function with respect to x, we get

$f'(x) = -3x^3 - 24x^2 - 45x$

Now take, -3x as common:

$= -3x(x^2 + 8x + 15)$

Factorise the expression inside the bracket, then we have:

$= -3x(x+5)(x+3)$

$f'(x) = 0$

$\Rightarrow x = -5, x = -3, x = 0$

Now, again differentiate the function:

$f''(x) = -9x^2 - 48x - 45$

Take -3 outside,

$= -3(3x^2 + 16x + 15)$

Now, substitue the value of x in the second derivative function.

$f''(0) = -45 < 0$. Hence, $x = 0$ is point of local maxima

$f''(-3) = 18 > 0$. Hence, $x = -3$ is point of local minima

$f''(-5) = -30 < 0$. Hence, $x = -5$ is point of local maxima.

Unit V: DIFFERENTIAL CALCULUS-II

Question 6: A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at a rate of 0.05 cm per second. Find the rate at which its area is increasing if the radius is 3.2 cm.

Solution: Let us assume that "r" be the radius of the given disc and "A" be the area, then the area is given as:

$A = \pi r^2$

By using the chain rule,

Then $dA/dt = 2\pi r(dr/dt)$

Thus, the approximate rate of increase of radius = $dr = (dr/dt)\,\Delta t = 0.05$ cm per second

Hence, the approximate rate of increase in area is:

$dA = (dA/dt)(\Delta t) = 2\pi r[(dr/dt)\,\Delta t\,]$

$= 2\pi\ (3.2)\ (0.05)$

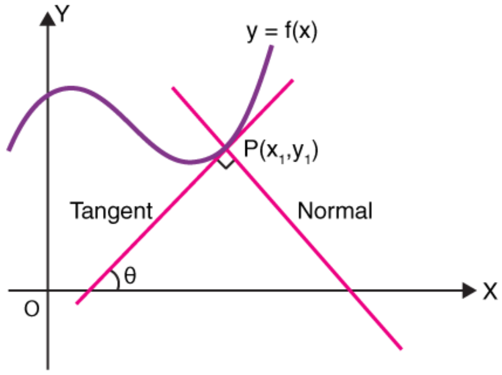
$= 0.320\pi\ cm^2$ per second.

Therefore, when r= 3.2 cm, then the area is increasing at a rate of $0.320\pi\ cm^2$/second.

Tangent and Normal To a Curve

A tangent is a line that touches the curve at a point and doesn't cross it, whereas normal is perpendicular to that tangent.
Let the tangent meet the curve at $P(x_1, y_1)$.

Unit V: DIFFERENTIAL CALCULUS-II



Now the straight-line equation which passes through a point having **slope m** could be written as;

$y - y_1 = m(x - x_1)$

We can see from the above equation, the **slope of the tangent** to the curve y = f(x) and at the point $P(x_1, y_1)$, it is given as **dy/dx at $P(x_1, y_1)$ = f'(x).** Therefore,

Slope of the normal : (-1/ slope of tangent at point P)= -(dx/dy)

Equation of the tangent to the curve at $P(x_1, y_1)$ can be written as:

$y - y_1 = f'(x_1)(x - x_1)$

Equation of normal to the curve is given by;

$y - y_1 = [-1/ f'(x_1)] (x - x_1)$

Or

$(y - y_1) f'(x_1) + (x-x_1) = 0$

Angle of intersection between two curves :-

Let m_1 be the slope of the tangent to the curve f(x) at (x_1, y_1).

$m_1 = df(x)/dx$ at (x_1, y_1)

Let m_2 be the slope of the tangent to the curve g(x) at (x_1, y_1).

$m_2 = dg(x)/dx$ at (x_1, y_1)

If θ is the acute angle of intersection between the given curves,

Then, $\tan\theta = |(m_1 - m_2)/(1 + m_1m_2)|$.

Condition for Orthogonal Curves

1. If $m_1m_2 = -1$, then $\theta = \pi/2$, which means the given curves cut orthogonally at the point (x_1, y_1) (meet at the right angle at the point (x_1, y_1)).

2. If $m_1 = 0$ and $m_2 = \infty$, then also the curves are orthogonal.

3. If $m_1 = m_2$, then the curves touch each other.

Example 1: The angle between the curves $xy = 2$ and $y^2 = 4x$ is

a. $\tan^{-1} 1/3$

b. $\tan^{-1} 2$

c. $\tan^{-1} 3$

d. $\tan^{-1} 2/3$

Solution: Given curves $xy = 2$ ..(i)

$y^2 = 4x$

From (i), $y = 2/x$

Substitute $y = 2/x$ in (ii)

We get $4/x^2 = 4x$

$\Rightarrow x^3 = 1$

$\Rightarrow x = 1$

Put $x = 1$ in (i), we get $y = 2$.

So, the point of intersection is (1, 2).

Slope of curve (i), $m_1 \Rightarrow x\, dy/dx + y$

$\Rightarrow dy/dx = -y/x$

$m_1 = (dy/dx)_{(1,2)} = -2$

Slope of curve (ii), $m_2 \Rightarrow 2y\, dy/dx = 4$

$\Rightarrow dy/dx = 2/y$

$m_2 = (dy/dx)_{(1,2)} = 2/2 = 1$

Angle between the given curves, $\tan\theta = |(m_1 - m_2)/(1 + m_1m_2)|$

$=|(-2 - 1)/(1 + -2)|$

$= 3$

So, $\theta = \tan^{-1} 3$.

Hence, option c is the answer.

Example 2: The line tangent to the curves $y^3-x^2y+5y-2x = 0$ and $x^2-x^3y^2+5x+2y = 0$ at the origin intersect at an angle θ equal to

(a) $\pi/6$

(b) $\pi/4$

(c) $\pi/3$

(d) π/2

Solution: Given curves $y^3-x^2y+5y-2x = 0$… (i)

$x^2-x^3y^2+5x+2y = 0$ …(ii)

Differentiate (i) w.r.t x

$3y^2 (dy/dx) – 2xy – x^2 (dy/dx) + 5 (dy/dx) – 2 = 0$

$(dy/dx)(3y^2 – x^2 + 5) = 2 + 2xy$

$(dy/dx) = (2 + 2xy)/(3y^2 – x^2 + 5)$

$m_1 = (dy/dx)_{(0,0)} = 2/5$

Differentiate (ii) w.r.t x

$x^2-x^3y^2+5x+2y = 0$

$2x-3x^2y^2-2yx^3(dy/dx)+5+2(dy/dx) = 0$

$\Rightarrow (dy/dx)(2-2yx^3) = (-2x+3x^2y^2-5)$

$\Rightarrow dy/dx = (-2x+3x^2y^2-5)/(2-2yx^3)$

$m_2 = (dy/dx)_{(0,0)} = -5/2$

$m_1m_2 = (2/5)(-5/2) = -1$

So $\theta = \pi/2$

Hence, option d is the answer.

Example 3: Find the equation of a tangent to the curve y = (x-7)/[(x-2)(x-3)] at the point where it cuts the x-axis.

Solution: As the point cut at the x-axis, then y=0. Hence, the equation of the curve, if y=0, then the value of x is 7. (i.e., x=7). Hence, the curve cuts the x-axis at (7,0)

Now, differentiate the equation of the curve with respect to x, we get

dy/dx = [(1-y)(2x-5)] / [(x-2)((x-3)]

$dy/dx]_{(7,\ 0)}$ = (1-0)/[(5)(4)] = 1/20

Hence, the slope of the tangent line at (7, 0) is 1/20.

Therefore, the equation of the tangent at (7, 0) is

Y-0 = (1/20)(x-7)

20y-x+7 = 0.

Example 4: Find the equation of tangent and normal to the curve $x^{(2/3)}+ y^{(2/3)} = 2$ at (1, 1)

Solution: Given curve: $x^{(2/3)}+ y^{(2/3)} = 2$

Finding Equation of Tangent:

Now, differentiate the curve with respect to x, we get

$(2/3)x^{(-1/3)} + (2/3)y^{(-1/3)}\, dy/dx = 0$

The above equation can be written as:

$dy/dx = -[y/x]^{1/3}$

Hence, the slope of the tangent at the point (1, 1) is $dy/dx]_{(1,1)} = -1$

Now, substituting the slope value in the tangent equation, we get

Equation of tangent at (1, 1) is

$y-1 = -1(x-1)$

$y+x-2 = 0$

Thus, the equation of tangent to the curve at (1, 1) is $y+x-2 = 0$

Finding Equation of Normal:

The slope of the normal at the point (1, 1) is

= -1/slope of the tangent at (1, 1)

$= -1/ -1$

$=1$

Therefore, the slope of the normal is 1.

Hence, the equation of the normal is

$y-1 = 1(x-1)$

$y-x = 0$

Therefore, the equation of the normal to the curve at (1, 1) is $y-x = 0$

Increasing and Decreasing Functions :-

To find that a given function is increasing or decreasing or constant, say in a graph, we use derivatives. If f is a function which is continuous in [p, q] and differentiable in the open interval (p, q), then,

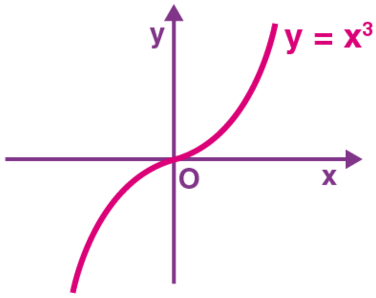
- f is increasing at [p, q] if $f'(x) > 0$ for each $x \in (p, q)$
- f is decreasing at [p, q] if $f'(x) < 0$ for each $x \in (p, q)$
- f is constant function in [p, q], if $f'(x)=0$ for each $x \in (p, q)$

Example: Check whether $y = x^3$ is an increasing or decreasing function.

Solution: $\frac{dy}{dx} = 3x^2 \geq 0$

So, it is an increasing function.

Graphical Representation:



Decreasing Function in Calculus

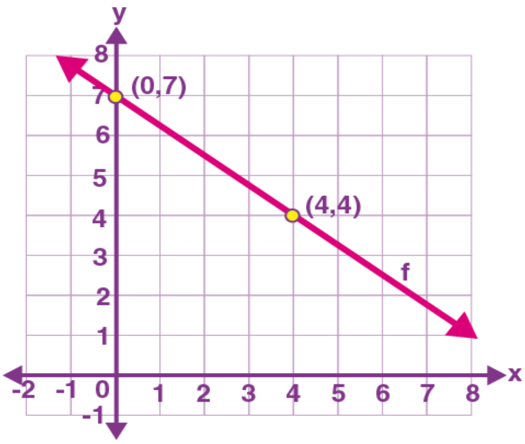
For a function, y = f(x) to be monotonically decreasing (dy/dx) ≤ 0 for all such values of interval (a, b), and equality may hold for discrete values.

Example: Check whether the function y = -3x/4 + 7 is an increasing or decreasing function.

Differentiate the function with respect to x, and we get

$$\frac{dy}{dx} = -\frac{3}{4} \leq 0$$

So, we can say it is a decreasing function.



Question 1: Prove that f(x) = x – sin(x) is an increasing function.

Solution: f(x) = x – sin(x)

$\frac{dy}{dx} = 1 - \cos x$

dy/dx) ≥ 0 as cos(x) having a value in the interval [-1, 1] and (dy/dx) = 0 for the discrete values of x and do not form an interval.

Hence, we can include this function as a monotonically increasing function.

Question 2: Prove that f(x) = cos x is decreasing function in [0, π].

Solution: f(x) = cos x

f'(x) = – sin x

As sin x is positive in the first and second quadrants, i.e., sin x ≥ 0 in [0, π], so we can say that

(dy/dx) = – (positive) = negative ≤ 0

$$\frac{dy}{dx} \leq 0$$

So, function f(x) = cos x is decreasing in [0, π].

Question 3: Find the value of "a" if the function $x^3 – 6x^2 + ax$ is increasing for all the values of x.

Solution:

Here $f(x) = x^3 – 6x^2 + ax$

$\Rightarrow f'(x) = 3x^2 – 12x + a$

For a function to be increasing $f'(x) > 0$

So, $3x^2 – 12x + a > 0$

$\Rightarrow 3(x^2 – 4x + a/3) > 0$

$\Rightarrow x^2 – 4x + a/3 > 0$

$\Rightarrow (x – 2)^2 – 2^2 + a/3 > 0$

$\Rightarrow (3(x – 2)^2 – 12 + a)/3 > 0$

$\Rightarrow 3(x – 2)^2 – 12 + a > 0$

We know that $3(x – 2)^2$ can't be negative and having the minimum value 0 at x = 2

The minimum value of f'(x) is at x = 2

min. f'(x) = -12 + a

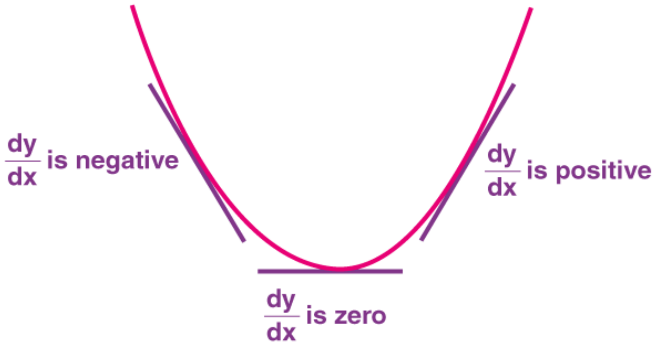
So for f'(x) > 0

$\Rightarrow -12 + a > 0$

$\Rightarrow a > 12.$

Unit V: DIFFERENTIAL CALCULUS-II

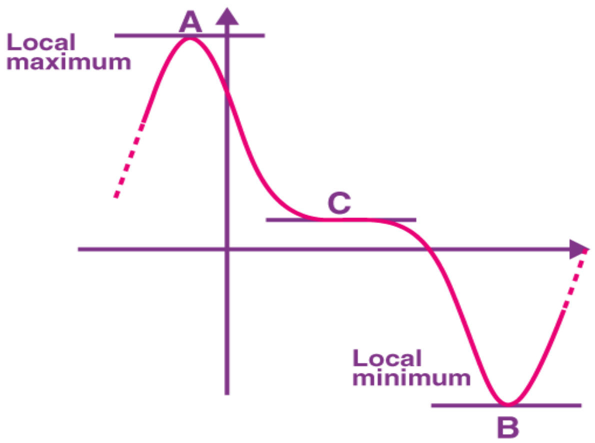
Maxima and Minima:-



Let f be a function defined on an open interval I.
Let f be continuous at a critical point c in I.
If f'(x) does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point is called a point of inflection.

Stationary Points vs Turning Points

Stationary points are the points where the slope of the graph becomes zero. In other words, the tangent of the function becomes horizontal, and dy/dx = 0. All the stationary points, A, B and C, are given in the figure shown below. And the points in which the function changes its path, if it was going upward; it will go downward and vice versa, i.e., points A and B are turning points since the curve changes its path. But point C is not a turning point, although the graph is flat for a short period of time but continues to go down from left to right.



Unit V: DIFFERENTIAL CALCULUS-II

Derivative Tests

The derivative test helps to find the maxima and minima of any function. Usually, the first-order derivative and second-order derivative tests are used. Let us have a look in detail.

First Order Derivative Test

Let f be the function defined in an open interval I. And f be continuous at critical point c in I such that f'(c) = 0.

1. If f'(x) changes sign from positive to negative as x increases through point c, then c is the point of local maxima, and the f(c) is the maximum value.
2. If f'(x) changes sign from negative to positive as x increases through point c, then c is the point of local minima, and the f(c) is the minimum value.
3. If f'(x) doesn't change sign as x increases through c, then c is neither a point of local nor a point of local maxima. It will be called the point of inflection.

Second Derivative Test

Let f be the function defined on an interval I, and it is two times differentiable at c.

i. x = c will be point of local maxima if f'(c) = 0 and f"(c)<0. Then, f(c) will be having local maximum value.

ii. x = c will be point of local minima if f'(c) = 0 and f"(c) > 0. Then, f(c) will be having local minimum value.

iii. When both f'(c) = 0 and f"(c) = 0, the test fails, and the first derivative test will give you the value of local maxima and minima.

Properties of maxima and minima

1. If f(x) is a continuous function in its domain, then at least one maximum or one minimum should lie between equal values of f(x).
2. Maxima and minima occur alternately, i.e., between two minima, there is one maxima and vice versa.
3. If f(x) tends to infinity as x tends to a or b and f'(x) = 0 only for one value x, i.e., c between a and b, then f(c) is the minimum and the least value. If

f(x) tends to $-\infty$ as x tends to a or b, then f(c) is the maximum and the highest value.

Question 1: Find the turning points of the function $y = 4x^3 + 12x^2 + 12x + 10$.

Answer: For turning points $dy/dx = 0$.

$dy/dx = 12x^2 + 24x + 12 = 0$

$\Rightarrow 3x^2 + 6x + 3 = 0$

$\Rightarrow (x + 1)(3x + 1) = 0$

$\Rightarrow x= -1$ and $x = (-1)/3$

Second derivative test:

At $x = -1$:

$d^2 y/dx^2 = 24x + 24 = 24(-1) + 24 = -24 + 24 = 0$.

Hence $x = -1$ is the point of inflection, it is a non-turning point.

At $x = (-1)/3$:

$d^2y/dx^2 = 24x + 24 = 24((-1)/3) + 24 = -8 + 24 = 16$.

Hence $x = (-1)/3$ is a point of minima, it is a turning point.

Question 2: Find the local maxima and minima of the function $f(x) = 3x^4 + 4x^3 – 12x^2 + 12$.

Answer:

For stationary points, f'(x) = 0.

$f'(x)= 12x^3 + 12x^2 – 24x = 0$

$\Rightarrow 12x(x^2 + x – 2) = 0$

$\Rightarrow 12x(x – 1)(x + 2) = 0$

=> Hence, $x = 0$, $x = 1$ and $x = -2$

Second derivative test:

$f''(x) = 36x^2 + 24x – 24$

$f''(x) = 12(3x^2 + 2x – 2)$

At $x = -2$

$f''(-2) = 12(3(-2)^2 + 2(-2) – 2) = 12 (12 – 4 – 2) = 12(6) = 72 > 0$

At $x = 0$

$f''(0) = 12(3(0)^2 + 2(0) – 2) = 12(-2) = -24 < 0$

At $x = 1$

$f''(1) = 12(3(1)^2 + 2(1) – 2) = 12(3+2-2) = 12(3) = 36 > 0$

Therefore, by the second derivative test, x=0 is the point of local maxima, while x = -2 and x = 1 are the points of local minima.

Question 3: Prove that the radius of the right circular cylinder of the greatest curved surface area, which can be inscribed in a given cone, is half of that of the cone.

Answer: Let r and h be the radius and height of the right circular cylinder inscribed in a given cone of radius R and height H. Let S be the curved surface area of the cylinder.

$S = 2\pi rh$

$h = H(R – r)/R$

So $S = 2\pi rH(R – r)/R$

$= (2\pi H/R)(rR – r^2)$

Differentiate w.r.t.r

$dS/dr = (2\pi H/R)(R – 2r)$

For maxima or minima,

$dS/dr =0$

=> $(2\pi H/R)(R – 2r) = 0$

=> $R – 2r = 0$

=> $R = 2r$

=> $r = R/2$

d2S/dr2 = $(2\pi H/R)(0 – 2)$

$= -4\pi H/R$ (negative)

So, for r = R/2, S is the maximum.

Question 4: A stone is thrown in the air. Its height at any time t is given by $h = -5t^2+10t+4$.

Find its maximum height.

Solution: Given $h = -5t^2+10t+4$

$dh/dt = -10t+10$

Now find when dh/dt = 0

$dh/dt = 0 \Rightarrow -10t+10 = 0$

$\Rightarrow -10t = -10$

$t = 10/10 = 1$

Height at $t = 1$ is given by $h = -5\times1^2+10\times1+4$

$= -5+10+4$

$= 9$

Hence, the maximum height is 9 m.

Question 5: Find the maxima and minima for $f(x) = 2x^3-21x^2+36x-15$

Solution: We have $f(x) = 2x^3-21x^2+36x-15$

$f'(x) = 6x^2-42x+36$

Now find the points where $f'(x) = 0$

$f'(x) = 0 \Rightarrow 6x^2-42x+36 = 0$

$\Rightarrow x^2-7x+6 = 0$

$\Rightarrow (x-6)(x-1) = 0$

$\Rightarrow$ $x = 6$ or $x = 1$ are the possible points of minima or maxima.

Let us test the function at each of these points.

$f''(x) = 12x-42$

At $x = 1$, $f''(1) = 12-42 = -30 <0$

Therefore $x = 1$ is a point of the local maximum.

The maximum value is $f(1) = 2-21+36-15 = 2$

At $x= 6$, $f''(x) = 12\times6-42 = 30> 0$

Therefore, $x = 6$ is a point of the local minimum.

The local minimum value is $f(6) = 2(6)^3-21(6)^2+36(6)-15$

$= 2\times216-21\times36+216-15$

$= 432-756+216-15 = -123$

VELOCITY:- It is the rate of change of displacement.

IF displacement from a fixed point be S after time t, then

Velocity V = dS/dt

ACCELERATION :- It is the rate of change of velocity.

$$a = \frac{dv}{dt}$$

Unit V: DIFFERENTIAL CALCULUS-II

Example 1: The position of a particle on a line is given by $s(t) = t^3 - 3t^2 - 6t + 5$, where t is measured in seconds and s is measured in feet. Find

a. The velocity of the particle at the end of 2 seconds.

b. The acceleration of the particle at the end of 2 seconds.

Part (a): The velocity of the particle is

$$v = s'(t) = 3t^2 - 6t - 6$$

$$\text{At } t = 2 \text{ seconds } s'(2) = 3(2)^2 - 6(2) - 6$$

$$s'(2) = -6 \text{ ft/sec}$$

Part (b): The acceleration of the particle is

$$a = v'(t) = s''(t) = 6t - 6$$

$$\text{At } t = 2 \text{ seconds } v'(2) = s''(2) = 6(2) - 6$$

$$v'(2) = s''(2) = 6 \text{ ft/sec}^2$$

Example 2: The formula $s(t) = -4.9t^2 + 49t + 15$ gives the height in meters of an object after it is thrown vertically upward from a point 15 meters above the ground at a velocity of 49 m/sec. How high above the ground will the object reach?

The velocity of the object will be zero at its highest point above the ground. That is, $v = s'(t) = 0$, where

$$v = s'(t) = -9.8t + 49$$

$$s'(t) = 0 \Rightarrow -9.8t + 49 = 0$$

$$-9.8t = -49$$

$$t = 5 \text{ seconds}$$

The height above the ground at 5 seconds is

$$s(5) = -4.9(5)^2 + 49(5) + 15$$

$$s(5) = 137.5 \text{ meters}$$

hence, the object will reach its highest point at 137.5 m above the ground.

Errors and approximation :-

Approximation:- Let y=f(x) and let δx be the little increment in x and δy be the corresponding increment in y, then

For small value of δx, (δx/ δy) = (dy/dx)

[but for small value δx the difference between δx δx abd dy/dx is very small and it can be decreased to ant extent.]

Small error:-

Let y=f(x) be any differential function.

Let δx be a small error in x and δy be the corresponding error in y, then

Small error in y =(dy/dx) x δx

Error in y = (dy/dx) x error in x

% error in x = (δx/x) x 100

% error in y = (δy/y) x 100

Unit V: DIFFERENTIAL CALCULUS-II

Ex. 1- Find the approximate change in the value of 1x2, when x changes from x = 2 to x = 2.002.

Solution :- Let $y=1/x^2$. Then $dy/dx=-2/x^3$.
$\therefore dy=(dy/dx)$at $x=2\times dx=(-2/2^3)\times 0.002=-0.002/4=-0.0005$.
Hence is decreased by 0.0005.

Ex.2 :- If $y = x^3 + 5$ and x changes from 3 to 2.99, then the approximate change is y is ____________________.

Solution:- Let $x = 3$ and $x + \Delta x = 2.99$.

$\therefore \Delta x = 2.99 - 3 = -0.01$

$y = x^3 + 5$ (Given)

Differentiating both sides with respect to x, we get

$dy/dx = 3x^2$

$\Rightarrow Dy/dx$ (at $x=3$) $=3 \times 3^2=27$

$\therefore \Delta y=(dy/dx)\Delta x$

$\therefore\Rightarrow \Delta y=27\times -0.01=-0.27$

Thus, the approximate change in y is -0.27.