Unit IV: Differential Calculus-I

FUNCTION :-

A function is defined as a relation between a set of inputs having one output each. In simple words, a function is a relationship between inputs where each input is related to exactly one output. Every function has a domain and codomain or range. A function is generally denoted by f(x) where x is the input. The general representation of a function is y = f(x).

Where x is an independent variable and y is a dependent variable.

An example of a simple function is $f(x) = x^2$. In this function, the function f(x) takes the value of "x" and then squares it. For instance, if x = 3, then f(3) = 9. A few more examples of functions are: f(x) = sin x, $f(x) = x^2 + 3$, f(x) = 1/x, f(x) = 2x + 3, etc.

DOMAIN : The **domain** is defined as the entire set of values possible for independent variables.

CODOMAIN:

In relations and functions, the codomain is the set of all possible outcomes of the given relation or function. Sometimes, the codomain is also equal to the range of the function. However, the range is the subset of the codomain.

RANGE : The Range is found after substituting the possible x- values to find the y-values.

Example 1: Find the domain and range of a function $f(x) = 3x^2 – 5$.

Solution: Given function:

$f(x) = 3x^2 – 5$

We know that the domain of a function is the set of input values for f, in which the function is real and defined.

The given function has no undefined values of x.

Thus, for the given function, the domain is the set of all real numbers.

Domain = [-∞, ∞]

Also, the range of a function comprises the set of values of a dependent variable for which the given function is defined.

Ley $y = 3x^2 – 5$

$3x^2 = y + 5$

$x^2 = (y + 5)/3$

$x = \sqrt{[(y + 5)/3]}$

Square root function will be defined for non-negative values.

So, $\sqrt{[(y + 5)/3]} \geq 0$

This is possible when y is greater than $y \geq -5$.

Hence, the range of f(x) is [-5, ∞).

Example 2: Find the domain and range of a function f(x) = (2x – 1)/(x + 4).

Solution: Given function is

f(x) = (2x – 1)/(x + 4)

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We know that the domain of a function is the set of input values for f, in which the function is real and defined.

The given function is not defined when x + 4 = 0, i.e. x = -4

So, the domain of given function is the set of all real number except -4.

i.e. Domain = (-∞, -4) U (-4, ∞)

Also, the range of a function comprises the set of values of a dependent variable for which the given function is defined.

Let y = (2x – 1)/(x + 4)

xy + 4y = 2x – 1

2x – xy = 4y + 1

x(2 – y) = 4y + 1

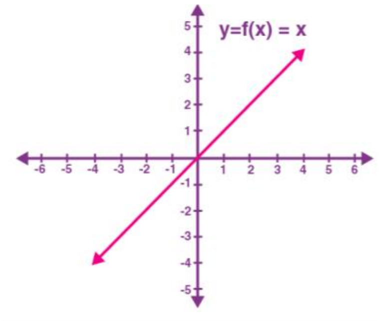
x = (4y + 1)/(2 – y)

This is defined only when y is not equal to 2.

Hence, the range of the given function is (-∞, 2) U (2, ∞).

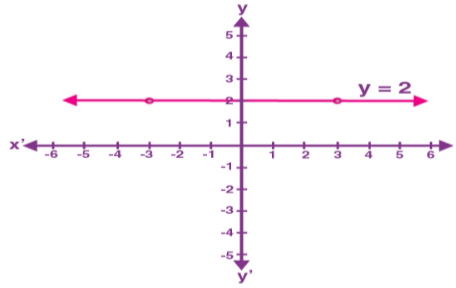
Identity Function

Suppose the real-valued function f : R → R by y = f(x) = x for each x ∈ R (i.e. the set of real numbers). Such a function is called the identity function. Also, the domain and range of this function f are R.



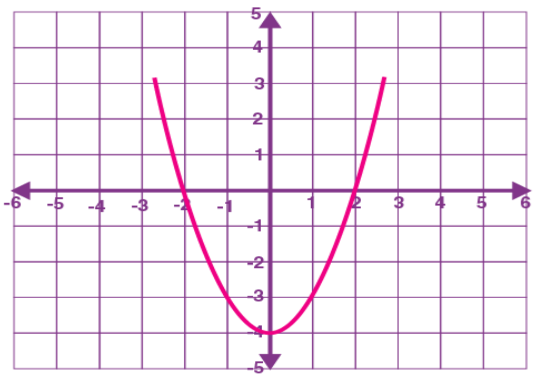
Constant Function

The function f: R → R by y = f (x) = c, x ∈ R where c is a constant and each x ∈ R is called a constant function. The domain of this function is R and its range is {c}. However, the graph of a constant function y = f(x) = 2 is given below.



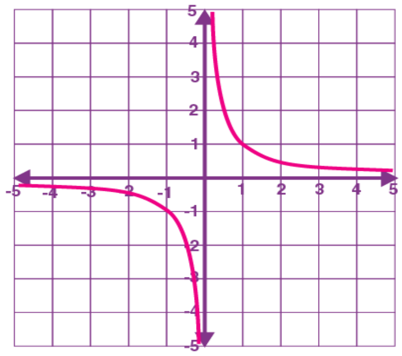
Polynomial Function

A function f : R → R is said to be polynomial function if for each x in R, y = f(x) = a_0 + a_1x + a_2x^2 + …+ a_nx^n, where n is a non-negative integer and a_0, a_1, a_2,…,a_n ∈ R. The graph of this type of function is a parabola. The graph of a certain polynomial function with degree 2 is given below:



Rational Functions

A function is of the form f(x)/g(x), where f(x) and g(x) are polynomial functions of x defined in a domain such that g(x) ≠ 0 is called a rational function. The example graph of a rational function is given below:



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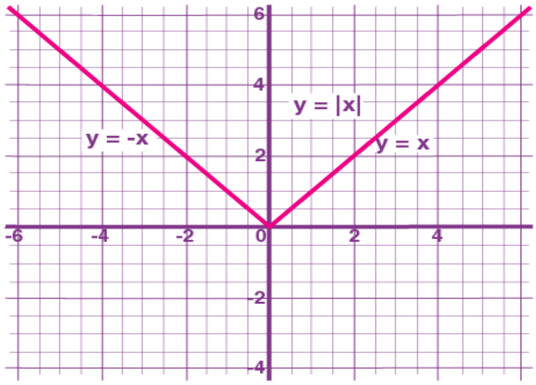
Modulus Function

The function f: R → R defined by y = f(x) = |x| for each x ∈ R is called the modulus function. The graph of a modulus function y = |x| is given below.

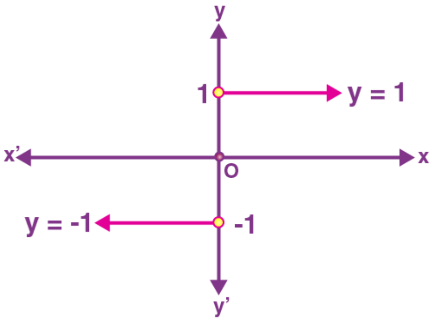


Signum Function

The function f: R → R defined by

$$f(x) = \begin{cases} -1; & x < 0 \\ 0 & x = 0 \\ 1; & x > 0 \end{cases}$$

is called the signum function. The domain of this function is R and the range is the set {–1, 0, 1}. The figure given below shows the graph of the signum function.



1. Which of the following relations are functions? Give reasons and also find the domain and range of the function.
(i) f = {(1, 3), (1, 5), (2, 3), (2, 5)}
(ii) g = {(2, 1), (5, 1), (8, 1), (11, 1)}
Solution: (i) f = {(1, 3), (1, 5), (2, 3), (2, 5)}
Here, the elements 1 and 2 have more than one f-images, namely 3 and 5.
Hence, f is not a function.

(ii) g = {(2, 1), (5, 1), (8, 1), (11, 1)}

Here, each first element of the ordered pair has a unique image which is the second coordinate.

Hence, g is a function.

Domain of g = {2, 5, 8, 11} and range of g = {1}

2. Let A = {1, 2} and B = {3, 6} and f and g be functions from A to B, defined by f(x) = 3x and g(x) = x^2 + 2. Show that f = g.

Solution: Since both f and g are defined from set A.

Therefore, dom(f) = dom(g)

Now, for the co-domain of f

f(1) = 3. 1 = 3

f(2) = 3. 2 = 6

Co-domain of f ={3, 6}

And the co-domain of g

g(1) = 1^2 + 2 = 3

g(2) = 2^2 + 2 = 6

Co-domain of g ={3, 6}

Co-domain of f = Co-domain of g

Also for each x ∈ A, f(x) = g(x)

Hence, f = g.

3. Find the domain and the range of the real function, f(x) = 1/(x + 3).

Solution: We have f(x) = 1/(x + 3)

Clearly, f is not defined for x = -3

Therefore, dom(f) = R – {-3}

Let y = f(x). Then,

y = 1/(x + 3) ⇒ x = (1/y) – 3 …….(i)

Clearly, (i) is not defined for y = 0

Therefore, range(f) = R – {0}

Limits:-

Definition: Let us consider a real-valued function "f" and the real number "c", the limit is normally defined as

$$\lim_{x \to c} f(x) = L$$

It is read as "the limit of f of x, as x approaches c equals L". The "lim" shows limit, and fact that function f(x) approaches the limit L as x approaches c is described by the right arrow.

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The **right-hand limit** of a function is the value the function approaches when the variable approaches its limit from the right.

This can be written as
$\lim_{x\to a} f(x) = A^+$.

The **left-hand limit** of a function is the value the function approaches when the variable approaches its limit from the left.

This can be written as $\lim_{x\to a} f(x) = A^-$

NOTE:- The limit of a function exists if and only if the left-hand limit is equal to the right-hand limit. $\lim_{x\to a^{-1}} f(x) = \lim_{x\to a^+} f(x) = L$

NOTE: Limits of the form $0/0, \infty/\infty, \infty-\infty, \infty\times 0, 1^\infty$,$0^0$ are called indeterminate forms.

Properties of Limits:-

The following is the list of properties of limits.

We assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and c is a constant. Then,

1.
$$\lim_{x\to a}[c.f(x)] = c\lim_{x\to a} f(x)$$

2. $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$
4. $\lim_{x\to a}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a} c = c$
6. $\lim_{x\to a} xn = an$

Example 1: To Compute To Compute $\lim_{x\to -4}(\mathbf{5x^2 + 8x - 3})$

Solution: First, use property 2 to divide the limit into three separate limits. Then use property 1 to bring the constants out of the first two. This gives,

$$\lim_{x\to -4}(5x^2 + 8x - 3) = \lim_{x\to -4}(5x^2) + \lim_{x\to -4}(8x) - \lim_{x\to -4}(3)$$
$$= 5(-4)^2 + 8(-4) - 3$$
$$= 80 - 32 - 3 = 45$$

Example 2: To Compute $\lim_{x\to 6}[\frac{(x-3)(x-2)}{x-4}]$

Solution: Given $\lim_{x\to 6}[\frac{(x-3)(x-2)}{x-4}]$

$$= \left[\frac{\lim_{x\to 6}(x-3)\lim_{x\to 6}(x-2)}{\lim_{x\to 6}(x-4)}\right]$$

$$= \left[\frac{(6-3)(6-2)}{(6-4)}\right]$$

$$= \left[\frac{(3)(4)}{(2)}\right] = 6$$

Example 3: Find the limit $\lim_{n\to\infty}$ [1/n^2 + 2/n^2 + … + n/n^2]

Solution: $\lim_{n\to\infty}$ [1/n^2 + 2/n^2 + … + n/n^2]

= $\lim_{n\to\infty}$ [1+2+3+…. +n]/ n^2

= $\lim_{n\to\infty}$ [(n / 2) * (n+1)] / n^2

= ½ $\lim_{n\to\infty}$ (n+1) / n

= ½ $\lim_{n\to\infty}$ (1 + 1/n)

= -½

Example 4: Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{x^2+a^2}-\sqrt{x^2+b^2}}{\sqrt{x^2+c^2}-\sqrt{x^2+d^2}}$$

Solution:

$$\lim_{x\to\infty} \frac{(a^2-b^2)}{(c^2-d^2)} \frac{\left[\sqrt{1+\frac{c^2}{x^2}}+\sqrt{1+\frac{d^2}{x^2}}\right]}{\left[\sqrt{1+\frac{a^2}{x^2}}+\sqrt{1+\frac{b^2}{x^2}}\right]} = \frac{a^2-b^2}{c^2-d^2}.$$

Special Rules:

1. $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{(n-1)}$, for all real values of n.
2. $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$
3. $\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1$
4. $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$
5. $\lim_{\theta\to 0}\cos\theta = 1$
6. $\lim_{x\to 0}e^x = 1$
7. $\lim_{x\to 0}\frac{e^x-1}{x} = 1$
8. $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$

9. $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$

10. $\lim_{x\to 0}\frac{a^x-1}{x} = \log a$

Example 1: Find $\lim_{x\to\infty} \sin x/x$.

Solution: Let x = 1/y or y = 1/x, so that $x \to \infty \Rightarrow y \to 0$

$\therefore \lim_{x\to\infty} (\sin x / x) = \lim_{y\to 0} (y \,.\, \sin (1 / y))$

$= \lim_{y\to 0} y \,.\, \lim_{y\to 0} \sin (1 / y)$

= 0

Example 2: Find $\lim_{x\to 0} \sin (\pi \cos^2 x) / x^2$

Solution:

$$\lim_{x\to 0}\left(\frac{\cos(\pi\cos^2 x).\,\pi.2\cos x(-\sin x)}{2x}\right)$$

$$= \lim_{x\to 0}\pi\cos(\pi\cos^2 x).\cos x.\left(\frac{-\sin x}{x}\right)$$

= π (-1).1.(-1) = π

Example 3: let f: R→R be such that f (1) = 3 and f'(1) = 6. Then find the value of $\lim_{x\to 0} [f (1+x) / f (1)]^{1/x}$.

Solution: Let $y = [f (1 + x) / f (1)]^{1/x}$

So, log y = 1/x [log f (1 + x) – log f (1)]

So, $\lim_{x\to 0} \log y = \lim_{x\to 0} [1 / f (1 + x) \,.\, f'(1 + x)]$

= f'(1) / f(1)

= 6/3

$\log (\lim_{x\to 0} y) = 2$

$\lim_{x\to 0} y = e^2$

Example 4: Evaluate: $\lim_{x\to 0}\frac{\sin ax}{bx}$

Solution:

$$\lim_{x\to 0}\frac{\sin ax}{bx}$$

Multiplying and dividing the function by "ax",

$$= \lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{ax}{bx}$$

$$= \lim_{x\to 0}\frac{\sin ax}{ax}\times\frac{a}{b}$$

Now, take the constant term out of the limit.

$$= \frac{a}{b} \times \lim_{x \to 0} \frac{\sin ax}{ax}$$

As we know,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

So, replacing “x” with “ax”,

$$\frac{a}{b} \times \lim_{ax \to 0} \frac{\sin ax}{ax} = \frac{a}{b} \times 1 = \frac{a}{b}$$

Therefore,

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

Example 5: Find the value of $\lim_{x \to \frac{\pi}{2}} \frac{\cos 2x}{\sin x}$

Solution: We know that, $\lim_{x \to a} \sin x = \sin a$ and $\lim_{x \to a} \cos x = \cos a$

Now,

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos 2x}{\sin x} = \frac{\lim_{x \to \frac{\pi}{2}} \cos 2x}{\lim_{x \to \frac{\pi}{2}} \sin x} = \frac{\cos 2(\frac{\pi}{2})}{\sin \frac{\pi}{2}}$$

= cos π/ (sin π/2)

= -1/1

= -1

Therefore,

$$\lim_{x \to \frac{\pi}{2}} \frac{\cos 2x}{\sin x} = -1$$

Example 6:

$$\lim_{x \to 0} \left(\frac{\sqrt{1 - \cos 2x}}{\sqrt{2}x}\right)$$

Solution: $\mathbf{If} \lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x) \mathbf{thenthelimitdoesnotexist}$

$$\lim_{x\to 0+}\left(\frac{\sqrt{1-\cos(2x)}}{\sqrt{2}x}\right)$$

$$\lim_{x\to a}[c\cdot f(x)] = c\cdot\lim_{x\to a}f(x)$$

$$=\frac{1}{\sqrt{2}}\cdot\lim_{x\to 0+}\left(\frac{\sqrt{1-\cos(2x)}}{x}\right)$$

ApplyL'Hospital'sRule

$$=\frac{1}{\sqrt{2}}\cdot\lim_{x\to 0+}\left(\frac{\frac{\sin(2x)}{\sqrt{1-\cos(2x)}}}{1}\right)$$

$$=\frac{1}{\sqrt{2}}\cdot\lim_{x\to 0+}\left(\frac{\sin(2x)}{\sqrt{1-\cos(2x)}}\right)$$

ApplyL'Hospital'sRule

$$=\frac{1}{\sqrt{2}}\cdot\lim_{x\to 0+}\left(\frac{\cos(2x)\cdot 2}{\frac{\sin(2x)}{\sqrt{1-\cos(2x)}}}\right)$$

$$\text{Simplify}\frac{\cos(2x)\cdot 2}{\frac{\sin(2x)}{\sqrt{1-\cos(2x)}}}$$

$$=\frac{1}{\sqrt{2}}\cdot\lim_{x\to 0+}\left(2\cot(2x)\sqrt{1-\cos(2x)}\right)$$

$$=\frac{1}{\sqrt{2}}\cdot 2\cdot\lim_{x\to 0+}\left(\cot(2x)\sqrt{1-\cos(2x)}\right)$$

Multiplybytheconjugateof $1-\cos(2x)$

$$=\frac{(1-\cos(2x))(1+\cos(2x))}{1+\cos(2x)}$$

Expand $(1-\cos(2x))(1+\cos(2x))$

$$=\frac{\sin^2(2x)}{1+\cos(2x)}$$

$$=\frac{1}{\sqrt{2}}\cdot 2\cdot\lim_{x\to 0+}\left(\cot(2x)\sqrt{\frac{\sin^2(2x)}{1+\cos(2x)}}\right)$$

$$=\frac{1}{\sqrt{2}}\cdot 2\cdot\lim_{x\to 0+}\left(\cos(2x)\sqrt{\frac{1}{\cos(2x)+1}}\right)$$

Pluginthevalue $x=0$

$$=\frac{1}{\sqrt{2}}\cdot 2\cos(2\cdot 0)\sqrt{\frac{1}{\cos(2\cdot 0)+1}}$$

$$\text{Simplify}\frac{1}{\sqrt{2}}\cdot 2\cos(2\cdot 0)\sqrt{\frac{1}{\cos(2\cdot 0)+1}}$$

$$=1$$

$$\text{Similarly}\lim_{x\to 0-}\left(\frac{\sqrt{1-\cos(2x)}}{\sqrt{2}x}\right)=-1$$

$$=\text{diverges}$$

Continuity:-

Definition:

A function is said to be continuous in a given interval if there is no break in the graph of the function in the entire interval range. Assume that “f” be a real function on a subset of the real numbers and “c” be a point in the domain of f. Then f is continuous at c if

$$\lim_{x \to c} f(x) = f(c)$$

In other words, if the left-hand limit, right-hand limit and the value of the function at x = c exist and are equal to each other, i.e.,

$$\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$$

then f is said to be continuous at x = c

Conditions for Continuity

- A function “f” is said to be continuous in an open interval (a, b) if it is continuous at every point in this interval.
- A function “f” is said to be continuous in a closed interval [a, b] if
  - f is continuous in (a, b)
  - $\lim_{x \to a^+} f(x) = f(a)$
  - $\lim_{x \to b^-} f(x) = f(b)$

Discontinuity Definition

The function “f” will be discontinuous at x = a in any of the following cases:

- f (a) is not defined.
- $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist but are not equal.
- $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are equal but not equal to f (a).

Types of Discontinuity

The four different types of discontinuities are:

- Removable Discontinuity
- Jump Discontinuity
- Infinite Discontinuity

Let’s discuss the different types of discontinuity in detail.

Removable Discontinuity

In removable discontinuity, a function which has well- defined two-sided limits at x = a, but either f(a) is not defined or f(a) is not equal to its limits. The removable discontinuity can be given as: $\boldsymbol{\lim_{x \to a} f(x) \neq f(a)}$

This type of discontinuity can be easily eliminated by redefining the function in such a way that $f(a) = \lim_{x \to a} f(x)$

Jump Discontinuity

Jump Discontinuity is a type of discontinuity, in which the left-hand limit and right-hand limit for a function x = a exists, but they are not equal to each other. The jump discontinuity can be represented as: $\boldsymbol{\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)}$

Infinite Discontinuity

In infinite discontinuity, the function diverges at x =a to give a discontinuous nature. It means that the function f(a) is not defined. Since the value of the function at x = a does not approach any finite value or tends to infinity, the limit of a function x → a are also not defined.

Example 1: Discuss the continuity of the function f(x) = sin x . cos x.

Solution: We know that sin x and cos x are the continuous function, the product of sin x and cos x should also be a continuous function.

Hence, f(x) = sin x . cos x is a continuous function.

Example 2:

Prove that the function f is defined by $f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at x $= 0$

Solution: Left hand limit at x = 0 is given by

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x\sin\frac{1}{x} = 0$$

$$\text{Similarly, } \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x\sin\frac{1}{x} = 0\text{ , }[f(0) = 0]$$

$$\text{Thus, } \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

Hence, the function f(x) is continuous at x =0.

Example 3: Find all the points of discontinuity of f **defined by** f(x)=|x|−|x+1|

Solution: Check continuity

Given function is **f(x)=|x|−|x+1|**

Let us consider the function

p(x)=|x|,q(x)=|x+1|

p(x)=|x| is modulus function. So, **p(x)** is continuous for all **x ∈ R**

q(x)=|x+1| is modulus function. So,**q(x)** is continuous for all **x ∈ R**

By Algebra of continuous functions,

If **p(x)** and **q(x)** all are continuous for all **x ∈ R** then **f(x)=p(x)−q(x)** is also continuous for all **x ∈ R**.

f(x)=|x|−|x+1| is continuous for all **x ∈ R**.

EXAMPLE 4: Find all points of discontinuity of f**, where** f **is defined by**

f(x) = {x+1,ifx≥1x2+1,ifx<1

Solution: The given function is **f(x)={x+1,ifx≥1x2+1,ifx<1**

The given function is defined at all the points of the real line.

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Let **c** be a point on the real line.

Case I: **c<1**, then **$f(c)=c^2+1$** and **$\lim_{x\to c} f(x)=\lim_{x\to c}(x^2+1)=c^2+1$ ∴ $\lim_{x\to c} f(x)=f(c)$**

Therefore, **f** is continuous at all points **x**, such that **x<1**

Case II : **c=1**, then **f(c)=f(1)=1+1=2**

The left hand limit of **f** at **x=1** is,

$\lim_{x\to 1} f(x)= \lim_{x\to 1} (x^2+1) = 1^2+1=2$

The right hand limit of **f** at **x=1** is,

$\lim_{x\to 1} f(x)=\lim_{x\to 1}(x+1)=1+1=2$

∴ $\lim_{x\to 1} f(x)=f(1)$

Therefore, **f** is continuous at **x=1**

Case III : **c>1**, then **f(c)=c+1**

$\lim_{x\to c} f(x)=\lim_{x\to c}(x+1)=c+1$

∴ $\lim_{x\to c} f(x)=f(c)$

Therefore, **f** is continuous at all points **x**, such that **x>1**

Hence, the given function **f** has no point of discontinuity.

Differentiation:-

Let y = f(x) be a function of x. Then, the rate of change of "y" per unit change in "x" is given by:

dy / dx

If the function f(x) undergoes an infinitesimal change of 'h' near to any point 'x', then the derivative of the function is defined as

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Derivative of Function As Limits

If we are given with real valued function (f) and x is a point in its domain of definition, then the derivative of function, f, is given by:

$f'(a) = \lim_{h\to 0}[f(x + h) – f(x)]/h$

provided this limit exists.

Notations:-

When a function is denoted as y = f(x), the derivative is indicated by the following notations.

1. **D(y) or D[f(x)]** is called Euler's notation.
2. **dy/dx** is called Leibniz's notation.

3. **F'(x)** is called Lagrange's notation.

Example: Find the derivative of f(x) = 2x, at x =3.

Solution: By using the above formulas, we can find,

$f'(3) = \lim_{h\to 0} [f(3 + h) - f(3)]/h = \lim_{h\to 0}[2(3 + h) - 2(3)]/h$

$f'(3) = \lim_{h\to 0} [6 + 2h - 6]/h$

$f'(3) = \lim_{h\to 0} 2h/h$

$f'(3) = \lim_{h\to 0} 2 = 2$

Differentiate the following functions from the first principles:

1. e^{-x}

Solution:

We have to find the derivative of e^{-x} with the first principle method,

So let $f(x) = e^{-x}$

By using the first principle formula, we get,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{e^{-(x+h)}-e^{-x}}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{e^{-x}(e^{-h}-1)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{e^{-x}(e^{-h}-1)(-1)}{h(-1)}$$

[By using $\lim_{x\to 0} \frac{e^{x}-1}{x} = 1$]

$$f'(x) = -e^{-x}$$

2 . $e^{\cos x}$

Solution: We have to find the derivative of $e^{\cos x}$ with the first principle method,

So, let $f(x) = e^{\cos x}$

By using the first principle formula, we get,

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{\cos(x+h)}-e^{\cos x}}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{\cos x}(e^{\cos(x+h)-\cos x}-1)}{h}$

$f'(x) = \lim_{h\to 0} \frac{e^{\cos x}(e^{\cos(x+h)-\cos x}-1)}{\cos(x+h)-\cos x} \frac{\cos(x+h)-\cos x}{h}$

[By using $\lim_{x\to 0} \frac{e^x-1}{x} = 1$]

$f'(x) = \lim_{h\to 0} e^{\cos x} \frac{\cos(x+h)-\cos x}{h}$

Now by using cos (x + h) = cos x cos h – sin x sin h

$f'(x) = \lim_{h\to 0} e^{\cos x} \frac{\cos x \cos h-\sin x \sin h-\cos x}{h}$

$f'(x) = \lim_{h\to 0} e^{\cos x} [\frac{\cos x(\cos h-1)}{h} - \frac{\sin x \sin h}{h}]$

[By using $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and $\cos 2x = 1-2\sin^2 x$]

$f'(x) = \lim_{h\to 0} e^{\cos x} [\frac{\cos x\left(-2\sin^2\frac{h}{2}\right)(\frac{h}{4})}{h(\frac{h}{4})} - \sin x]$

Differentiate the following functions with respect to x:

1. **Sin (3x + 5)**

Solution: Given Sin (3x + 5)

Let y = sin (3x + 5)

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(3x+5)]$$

We know $\frac{d}{dx}(\sin x) = \cos x$

$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\frac{d}{dx}(3x+5)$ [Using chain rule]

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\left[\frac{d}{dx}(3x) + \frac{d}{dx}(5)\right]$$

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\left[3\frac{d}{dx}(x) + \frac{d}{dx}(5)\right]$$

However, $\frac{d}{dx}(x) = 1$ and derivative of a constant is 0.

$$\Rightarrow \frac{dy}{dx} = \cos(3x+5)\,[3 \times 1 + 0]$$

$$\therefore \frac{dy}{dx} = 3\cos(3x+5)$$

Thus, $\frac{d}{dx}[\sin(3x+5)] = 3\cos(3x+5)$

2. tan² x

Solution: Given $\tan^2 x$

Let $y = \tan^2 x$

On differentiating y with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^2 x)$$

We know $\frac{d}{dx}(x^n) = nx^{n-1}$

$\Rightarrow \frac{dy}{dx} = 2\tan^{2-1} x \frac{d}{dx}(\tan x)$ [Using chain rule]

$$\Rightarrow \frac{dy}{dx} = 2\tan x \frac{d}{dx}(\tan x)$$

However, $\frac{d}{dx}(\tan x) = \sec^2 x$

$$\Rightarrow \frac{dy}{dx} = 2\tan x\,(\sec^2 x)$$

$$\therefore \frac{dy}{dx} = 2\tan x \sec^2 x$$

Thus, $\frac{d}{dx}(\tan^2 x) = 2\tan x \sec^2 x$

Unit IV: Differential Calculus-I

Differentiation Formulas

The important Differentiation formulas for Trigonometric functions, exponential, function of function, logarithmic functions are given below in the table. Here, let us consider f(x) as a function and f'(x) is the derivative of the function.

1. If f(x) = tan (x), then f'(x) = $\sec^2 x$
2. If f(x) = cos (x), then f'(x) = -sin x
3. If f(x) = sin (x), then f'(x) = cos x
4. If f(x) = ln(x), then f'(x) = 1/x
5. If f(x) = e^x, then f'(x) = e^x
6. If f(x) = x^n, where n is any fraction or integer, then f'(x) = nx^{n-1}
7. If f(x) = k, where k is a constant, then f'(x) = 0
8. If f(x) = sec (x), then f'(x) = sec(x)tan(x)
9. If f(x) = cot (x), then f'(x) = $-\mathrm{cosec}^2 x$
10. If f(x) = cosec (x), then f'(x) = - cosec x.cot x

Exponential and logarithmic functions

11.
$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$$
$$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}, \text{where } a > 0, a \neq 1$$
$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$
$$\frac{d}{dx}(\ln_a u) = \frac{1}{u\ln a}\frac{du}{dx}, \text{ where } a > 0, a \neq 1$$

Derivatives of inverse trigonometric functions

12.
$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, -1 < u < 1$$
$$\frac{d}{dx}(\cos^{-1} u) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, -1 < u < 1$$
$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$$
$$\frac{d}{dx}\ cosec^{-1} u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} |u| > 1$$
$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} |u| > 1$$
$$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx}$$

Differentiation Rules

The basic differentiation rules that need to be followed are as follows:

- Sum and Difference Rule
- Product Rule
- Quotient Rule
- Chain Rule

Let us discuss all these rules here.

Sum or Difference Rule

If the function is the sum or difference of two functions, the derivative of the functions is the sum or difference of the individual functions, i.e.,

If f(x) = u(x) ± v(x)

then, f'(x) = u'(x) ± v'(x)

Product Rule

As per the product rule, if the function f(x) is product of two functions u(x) and v(x), the derivative of the function is,

If

$$f(x) = u(x) \times v(x)$$

then,

$$\mathbf{f}'(\mathbf{x}) = \mathbf{u}'(\mathbf{x}) \times \mathbf{v}(\mathbf{x}) + \mathbf{u}(\mathbf{x}) \times \mathbf{v}'(\mathbf{x})$$

Quotient rule

If the function f(x) is in the form of two functions [u(x)]/[v(x)], the derivative of the function is

If,

$$f(x) = \frac{u(x)}{v(x)}$$

then,

$$\mathbf{f}'(\mathbf{x}) = \frac{\mathbf{u}'(\mathbf{x}) \times \mathbf{v}(\mathbf{x}) - \mathbf{u}(\mathbf{x}) \times \mathbf{v}'(\mathbf{x})}{(\mathbf{v}(\mathbf{x}))^2}$$

Chain Rule

If a function y = f(x) = g(u) and if u = h(x), then the chain rule for differentiation is defined as,

$$\frac{d\mathbf{y}}{d\mathbf{x}} = \frac{d\mathbf{y}}{d\mathbf{u}} \times \frac{d\mathbf{u}}{d\mathbf{x}}$$

Q.1: Differentiate $f(x) = 6x^3 - 9x + 4$ with respect to x.

Solution: Given: $f(x) = 6x^3 - 9x + 4$

On differentiating both the sides w.r.t x, we get;

$f'(x) = (3)(6)x^2 - 9$

$f'(x) = 18x^2 - 9$

This is the final answer.

Unit IV: Differential Calculus-I

Q.2: Differentiate $y = x(3x^2 – 9)$

Solution: Given, $y = x(3x^2 – 9)$

$y = 3x^3 – 9x$

On differentiating both the sides we get,

$dy/dx = 9x^2 – 9$

PROBLEMS ON DIFFERENTIATION OF THE PRODUCT OF FUNCTIONS:-

Example 1: Simplify the expression: $y= x^2 \times x^5$

Solution: Given: $y= x^2 \times x^5$

We know that the product rule for the exponent is

$x^n \times x^m = x^{n+m.}$

By using the product rule, it can be written as:

$y = x^2 \times x^5 = x^{2+5}$

$y = x^7$

Hence, the simplified form of the expression, $y= x^2 \times x^5$ is x^7.

Example 2: Differentiate y = sin x cos x

Solution: Given: y = sin x cos x

dy/dx = d(sinx cos x)/dx

While differentiating, it becomes

dy/dx = (sin x) [d(cos x)/dx] + (cos x) [d(sin x)/dx]

Differentiate the terms, dy/dx = sin x (-sin x) + cos x (cos x)

$dy/dx = -\sin^{2.x} + \cos^2 x$

$dy/dx =\cos^2 x – \sin^2 x$

By using identity,

dy/dx = cos 2x

Therefore, dy/dx = cos 2x

Example 3: Differentiate the function: $(x^2 + 3)(5x + 4)$

Solution: Given function is: $(x^2 + 3)(5x + 4)$

Here $u = (x^2 + 3)$ and $v = (5x + 4)$

Using product rule,

$$\frac{d((x^2+3)(5x+4))}{dx}$$

$(x^2 + 3)\frac{d(5x+4)}{dx} + (5x + 4)\frac{d(x^2+3)}{dx}$

$= (x^2 + 3)\ 5 + (5x + 4)\ 2x$

$= 5x^2 + 15 + 10x^2 + 8x$

$= 15x^2 + 8x + 15$

Unit IV: Differential Calculus-I

Differentiation of Trigonometric Function :-

Derivatives of Trigonometric Functions Formulas

1. If f(x) = tan (x), then f'(x) = sec^2x
2. If f(x) = cos (x), then f'(x) = -sin x
3. If f(x) = sin (x), then f'(x) = cos x
4. If f(x) = ln(x), then f'(x) = 1/x
5. If f(x) = e^x, then f'(x) = e^x
6. If f(x) = x^n, where n is any fraction or integer, then f'(x) = nx^{n-1}
7. If f(x) = k, where k is a constant, then f'(x) = 0
8. If f(x) = sec (x), then f'(x) = sec(x)tan(x)
9. If f(x) = cot (x), then f'(x) = -$cosec^2x$
10. If f(x) = cosec (x), then f'(x) = - cosec x.cot x

Example 1: Find the derivative of f(x) = x sin x – $4x^2$.

Solution: Given

f(x) = x sin x – $4x^2$

(d/dx) f(x) = (d/dx) [x sin x – $4x^2$]

= (d/dx) (x sin x) – (d/dx)$4x^2$

Using the product and power rule of differentiation,

= x [(d/dx) sin x] + sin x [(d/dx) x] – 4 (d/dx)x^2

= x cos x + sin x (1) – 4 (2x)

= x cos x + sin x – 8x.

Example 2: Find the derivative of f(x) = tan 4x

Solution:

f(x) = tan 4x

⇒ f′(x) = (d/dx) [tan 4x]

By applying chain rule

f′(x) = (d/dx) [tan 4x](d/dx)[4x]

⇒ f′(x) = (sec^2 4x)(4)

Example 3: Find the derivative of f(x) = cos(x^2 +4)

Solution:

f(x) = cos(x^2 +4)

⇒ f′(x) = (d/dx) cos(x^2 +4)

By applying chain rule

f′(x) = (d/dx) [cos(x^2 +4)](d/dx)[x^2 +4]

⇒ f′(x) = -(2x)sin(x^2 +4)

Example 4: Find the derivative of y = cos x / ($4x^2$)

Solution:

y = cos x / ($4x^2$)

Applying quotient rule

y' = [(d/dx)cosx($4x^2$) – cosx (d/dx)($4x^2$)] / $(4x^2)^2$

⇒ y' = [(-sinx)($4x^2$) – cosx (8x)] / ($16x^4$)

⇒ y' = [-4x²sinx – 8xcosx] / (16x⁴)
⇒ y' = [-4x(xsinx + 2cosx)] / (16x⁴)
⇒ y' = – (x sinx + 2cosx) / (4x³)

Differentiation of Exponential Function :-

If f(x) = e^x, then f'(x) = e^x

Example 1: Find the derivative of log (log x), x > 1, with respect to x.

Solution:

$$\frac{d}{dx}y = \frac{d}{dx}log(log\ x)$$

Using the formula d/dx (log x) = 1/x,

$$= \frac{1}{log\ x}\ \frac{d}{dx}(log\ x)$$

$$= \frac{1}{log\ x}.\frac{1}{x}$$

Therefore, (d/dx) [log (log x)] = 1/(x log x).

Example 2: Evaluate the derivative of the function

$y = e^{x^4}$.

Solution: Given function is:

$y = e^{x^4}$

Differentiating with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{x^4}) = e^{x^4}\frac{d}{dx}(x^4) = e^{x^4}.4x^{4-1} = 4x^3e^{x^4}$$

DIFFERENTIATION OF THE FUNCTION OF A FUNCTION:-

Example 1: Find dy/dx for y = x sinx log x

Solution:

$\boldsymbol{y' = 1 \times \sin x \times \log x + x\cos x\log x + x\sin x \times \frac{1}{x} = \sin x\log x + x\cos x\log x + \sin x}$

Example 2: Find dy/dx for y = sin(x^2 + 1).

Solution: $\boldsymbol{y' = \cos(x^2 + 1) \times 2x}$

= 2x cos (x^2 + 1)

Example 3: Differentiate sin(3x+5)

Solution: Say, y = sin (3x+5)

dy/dx = d[sin(3x+5)]/dx

= cos (3x+5) d(3x+5)/dx [By chain rule]

= cos (3x+5) [3]

y' = 3 cos (3x+5)

d[sin(3x+5)]/dx = 3 cos (3x+5)

Unit IV: Differential Calculus-I

Example 4: Differentiate $\tan^2 x$.

Solution: Say, $y = \tan^2 x$

$dy/dx = d(\tan^2 x)/dx$

$= 2\tan^{2-1} x.\ d(\tan x)/dx$

$= 2\tan x \sec^2 x$

$d[\tan^2 x]/dx = 2\tan x \sec^2 x$

Example 5: Compute the derivative of $f(x) = \sin^2 x$.

Solution: $f(x) = \sin^2 x = \sin x \sin x$

$= d(\sin x)/dx.\ \sin x + \sin x.d(\sin x)/dx$

$= \cos x.\ \sin x + \sin x \cos x$

$= 2 \sin x \cos x$

Differentiation of Logarithmic Function :-

Q. 1: Find the value of dy/dx if, $y = e^{x^4}$

Solution: Given function: $y = e^{x^4}$

Taking natural logarithm of both the sides we get,

$\ln y = \ln e^{x^4}$

$\ln y = x^4 \ln e$

$\ln y = x^4$

Now, differentiating both the sides w.r.t we get,

$$\frac{1}{y}\frac{dy}{dx} = 4x^3$$

$$\Rightarrow \frac{dy}{dx} = y.4x^3$$

$$\Rightarrow \frac{dy}{dx} = e^{x^4} \times 4x^3$$

Q. 2: Find the value of $\frac{dy}{dx}$ if $y = 2x^{cosx}$

Solution: Given the function $y = 2x^{\cos x}$

Taking logarithm of both the sides, we get

$\log y = \log(2x^{\{\cos x\}})$

$$\Rightarrow logy = log2 + logx^{cosx}$$
$$(As\ log(mn) = logm + logn)$$
$$\Rightarrow logy = log2 + cosx \times logx$$
$$(As\ logm^n = nlogm)$$

Now, differentiating both the sides w.r.t by using the chain rule we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{cosx}{x} - (sinx)(logx)$$

Differentiation of Inverse Trigonometric Function :-

Formulas of Differentiation of Inverse Trigonometric Function :-

- $d/dx\,(\sin^{-1}x) = 1/\sqrt{1-x^2}$
- $d/dx(\cos^{-1}x) = -1/\sqrt{1-x^2}$
- $d/dx(\tan^{-1}x) = 1/(1+x^2)$
- $d/dx(\csc^{-1}x) = -1/(|x|\sqrt{x^2-1})$
- $d/dx(\sec^{-1}x) = 1/(|x|\sqrt{x^2-1})$
- $d/dx(\cot^{-1}x) = -1/(1+x^2)$

Example: Find the derivative of a function $\sin^{-1}(\frac{1-x^2}{1+x^2})$

Solution: Given y = $\sin^{-1}(\frac{1-x^2}{1+x^2})$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \times \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}} \times \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx} = \frac{1+x^2}{\sqrt{(1+x^4+2x^2)-(1+x^4-2x^2)}} \times \left(\frac{(1+x^2)(-2x)-(1-x^2)(2x))}{(1+x^2)^2}\right)$$

$$\frac{dy}{dx} = \frac{1+x^2}{\sqrt{4x^2}} \times \left(\frac{(-2x-2x^3-2x+2x^3)}{(1+x^2)^2}\right)$$

$$\frac{dy}{dx} = \frac{1+x^2}{2x} \times \left(\frac{-4x}{(1+x^2)^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

Example: Find the derivative of f(x) = $3\sin^{-1}x + 4\cos^{-1}x$

Solution: *f′(x) = (d/dx) [$3\sin^{-1}x + 4\cos^{-1}x$]*

⇒ f′(x) = (d/dx) [$3\sin^{-1}x$]+ (d/dx) [$4\cos^{-1}x$]

⇒ f′(x) = 3(d/dx) [$\sin^{-1}x$]+ 4(d/dx) [$\cos^{-1}x$]

⇒ f′(x) = 3[1 / √(1 – x^2)] + 4[-1 / √(1 – x^2)]

⇒ f′(x) = 3[1 / √(1 – x^2)] – 4[1 / √(1 – x^2)]

⇒ f'(x) = [1 / √(1 – x^2)] (3- 4)
⇒ f'(x) = -[1 / √(1 – x^2)]

Example : What is the derivative of $\sin^{-1}(2x^3)$?
Solution: By the inverse trig derivatives,
d/dx ($\sin^{-1}x$) = $1/\sqrt{1-x^2}$
Using this and also the chain rule,
d/dx ($\sin^{-1}(2x^3)$) = $1/\sqrt{1-(2x^3)^2}$ d/dx($2x^3$)
= $1/\sqrt{1-4x^6}$ ($6x^2$)
Answer: The derivative of $\sin^{-1}(2x^3)$ is $(6x^2)/\sqrt{1-4x^6}$.

Differentiation of Implicit Function :-

Implicit differentiation, also known as the **chain rule**, differentiate both sides of an equation with two given variables by considering one of the variables as a function of the second variable. In short, differentiate the given function with respect to x and solve for dy/dx. Let us have a look at some examples.

Example 1: If $x^2 + 2xy + y^3 = 4$, find dy/dx.
Solution: Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(4)$$

$$= 2x + 2x\frac{dy}{dx} + 2y + 3y^2.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2(x+y)}{(2x+3y^2)}$$

Example 2: Differentiate log sin x w.r.t $\sqrt{\cos x}$
Solution: Let u = log sin x and v = $\sqrt{\cos x}$

$$\frac{du}{dx} = \cot x \& \frac{dv}{dx} = \frac{-\sin x}{2\sqrt{\cos x}}$$

$$\frac{du}{dx} = \frac{du/dx}{dv/dx} = \frac{\cot x}{\frac{-\sin x}{2\sqrt{\cos x}}} = -2\sqrt{\cos x}\cot x \cos ec(x)$$

Example 3: Find $\frac{dy}{dx}$ if $x^4 + y^3 – 3x^2y = 0$.

Solution 3: The given function $x^4 + y^3 – 3x^2y = 0$ can be differentiated using the concept of implicit function differentiation. Therefore differentiating both the sides w.r.t. x, we get,

$$4x^3 + 3y^2\frac{dy}{dx} – 3(2xy + x^2\frac{dy}{dx}) = 0$$

$$\frac{dy}{dx}(3x^2 - 3y^2) = 4x^3 - 6xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 6xy}{3x^2 - 3y^2}$$

Example 4: Differentiate $x^2 + y^2 = 25$ implicitly.

Solution: Differentiating $x^2 + y^2 = 25$ with respect to x we get;

2x + 2y dy/dx = 0

2y dy/dx = -2x

dy/dx = -2x/2y

dy/dx = -x/y

Example 5: Differentiate $x^3 + y^2 = 16$ implicitly.

Solution: Differentiating $x^3 + y^2 = 16$ with respect to x we get;

$3x^2 + 2y$ dy/dx =0

2y dy/dx = $-3x^2$

dy/dx = $-3x^2/2y$