Unit III : TRIGONOMETRY

Sine Rule, **Sine Law** or **Sine Formula:-**

In general, the law of sines is defined as the ratio of side length to the sine of the opposite angle. It holds for all the three sides of a triangle respective of their sides and angles.

a/sinA = b/sinB = c/sinC.

Cosine rule or law of cosines or **Cosine Formula:-**

The cosine rule relates to the lengths of the sides of a triangle with any of its angles being a cosine angle. With the help of this rule, we can calculate the length of the side of a triangle or can find the measure of the angle between the sides.

- cos x = $(b^2 + c^2 - a^2)/2bc$
- cos y = $(a^2 + c^2 - b^2)/2ac$
- cos z = $(a^2 + b^2 - c^2)/2ab$

Projection formula:-

projection formulae to find out the sides and angles of the triangle ABC can be shown as follows:

a = b Cos C + c Cos B

b = c Cos A + a Cos C

c = a Cos B + b Cos A.

Tangent Formula)

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2};\ \tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2};\ \tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};\ \sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ac}};\ \sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};\ \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}};\ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{(s-b)(s-c)}{\Delta};\ \tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} = \frac{(s-c)(s-a)}{\Delta}$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} = \frac{(s-a)(s-b)}{\Delta}$$

Area of triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc} \quad \text{or} \quad \Delta = \frac{1}{2}bc\sin A$$

$$\sin B = \frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ac} \quad \text{or} \quad \Delta = \frac{1}{2}ac\sin B$$

$$\sin C = \frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{ab} \quad \text{or} \quad \Delta = \frac{1}{2}ab\sin C$$

where $2s = a + b + c =$ perimeter , $\Delta =$ area of a triangle.

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Example:

In any ΔABC, $\angle A = 30°$, $\angle C = 45°$, find $a : c$.

In ΔABC from sine rule

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\sin 30°} = \frac{c}{\sin 45°}$$

$$\frac{a}{c} = \frac{\sin 30°}{\sin 45°} = \frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = \frac{1}{2} \times \frac{\sqrt{2}}{1} = \frac{1}{\sqrt{2}}$$

$$a : c = 1 : \sqrt{2}$$

EXAMPLE:-

Prove $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

Solution : L.H.S. $= \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{1}{a}\cos A + \frac{1}{b}\cos B + \frac{1}{c}\cos C$

$$= \frac{1}{a}\frac{b^2+c^2-a^2}{2bc} + \frac{1}{b}\frac{c^2+a^2-b^2}{2ac} + \frac{1}{c}\frac{a^2+b^2-c^2}{2ab}$$

[putting the values of $\cos A$, $\cos B$ and $\cos C$]

$$= \frac{1}{2abc}[b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2]$$

$$= \frac{1}{2abc}[a^2+b^2+c^2] = \frac{a^2+b^2+c^2}{2abc} = \text{R.H.S.}$$

Proved.

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Example:-

: Prove $a \sin(B-C) + b \sin(C-A) + c \sin(A-B) = 0$

Solution : L.H.S. $= a\sin(B-C) + b\sin(C-A) + c\sin(A-B)$

$= K \sin A \sin(B-C) + K \sin B \sin(C-A) + K \sin C \sin(A-B)$

$$\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K\right]$$

$= K.[\sin A \sin(B-C) + \sin B \sin(C-A) + \sin C \sin(A-B)]$

$= K[\sin\{\pi-(B+C)\}\sin(B-C) + \sin\{\pi-(C+A)\}\sin(C-A)$
$+ \sin\{\pi-(A+B)\}\sin(A-B)]$

$[\because A+B+C=\pi]$

$= K[\sin(B+C)\sin(B-C) + \sin(C+A)\sin(C-A) + \sin(A+B)\sin(A-B)]$

$[\because \sin(\pi-\theta)=\theta]$

$= K[\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B]$

$[\because \sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta]$

$= K \times 0 = 0 =$ R.H.S. **Proved.**

EXAMPLE :-

If $\sin 2A + \sin 2B = \sin 2C$ for any ΔABC, prove $A = 90°$ or $B = 90°$

Solution : Given $\sin 2A + \sin 2B = \sin 2C$

or $2\sin\frac{2A+2B}{2}\cdot\cos\frac{2A-2B}{2} = 2\sin C\cos C$

or $\sin(A+B)\cdot\cos(A-B) = \sin C\cos C$

or $\sin(\pi-C)\cdot\cos(A-B) = \sin C\cos\{\pi-(A+B)\}$ $[\because A+B+C=\pi]$

or $\sin C\cos(A-B) = -\sin C\cos(A+B)$

or $\sin C[\cos(A-B) + \cos(A+B)] = 0$

or $\sin C[2\cos A\cdot\cos B] = 0$

$\therefore$ $\sin C = 0$ or $2\cos A\cdot\cos B = 0$

But $\sin C = 0 \Rightarrow C = 0$, which is not possible for a Δ.

$\therefore$ $2\cos A\cdot\cos B = 0$ or $\cos A\cdot\cos B = 0$

$\therefore$ $\cos A = 0$ or $\cos B = 0$

But $\cos A = 0 \Rightarrow A = \frac{\pi}{2} = 90°$ and $\cos B = 0 \Rightarrow B = \frac{\pi}{2} = 90°$ $[\because \cos 90° = 0]$

$\therefore$ $A = 90°$ or $B = 90°$ **Proved.**

EXAMPLE:-

If the sides of a triangle ABC are in H.P. prove

$$\text{cosec}^2\frac{A}{2}, \text{cosec}^2\frac{B}{2}, \text{cosec}^2\frac{C}{2} \text{ are in A.P.}$$

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Solution : Let the sides of Δ be a, b, c. Now a, b, c are in H.P. [given]

$\therefore \quad \frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ will be in A.P.

$\Rightarrow \quad \frac{s}{a}, \frac{s}{b}$ and $\frac{s}{c}$ will be in A.P. $\Rightarrow \frac{s}{a}-1, \frac{s}{b}-1$ or $\frac{s}{c}-1$ in A.P.

$\Rightarrow \quad \frac{s-a}{a}, \frac{s-b}{b}$ and $\frac{s-c}{c}$ are also in A.P.

$\Rightarrow \quad \frac{s-a}{a} \times \frac{abc}{(s-a)(s-b)(s-c)}, \frac{s-b}{b} \times \frac{abc}{(s-a)(s-b)(s-c)}$

and $\quad \frac{s-c}{c} \times \frac{abc}{(s-a)(s-b)(s-c)}$ will be in A.P.

$\Rightarrow \quad \frac{bc}{(s-b)(s-c)}, \frac{ac}{(s-a)(s-c)}$ and $\frac{ab}{(s-a)(s-b)}$ will be in A.P.

$\Rightarrow \quad \operatorname{cosec}^2 \frac{A}{2}, \operatorname{cosec}^2 \frac{B}{2}$ and $\operatorname{cosec}^2 \frac{C}{2}$ will be in A.P. [$\because \operatorname{cosec} \frac{A}{2} = \sqrt{\frac{bc}{(s-b)(s-c)}}$ etc.]

INVERSE CIRCULAR FUNCTION:-

1. If $\sin\theta = x$ then $\theta = \sin^{-1} x$
2. If $\sin^{-1} x = \theta$ then $\sin\theta = x$
3. $\sin^{-1} x = \operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$; $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right)$ etc.
4. $\theta = \sin^{-1}(\sin\theta) = \cos^{-1}(\cos\theta) = \tan^{-1}(\tan\theta) = \cot^{-1}(\cot\theta)$ etc.
5. $x = \sin(\sin^{-1} x) = \cos(\cos^{-1} x) = \tan(\tan^{-1} x) = \cot(\cot^{-1} x)$ etc.
6. $\sin^{-1}(-x) = -\sin^{-1} x$; $\cos^{-1}(-x) = \pi - \cos^{-1} x$
   $\tan^{-1}(-x) = -\tan^{-1} x$; $\cot^{-1}(-x) = \pi - \cot^{-1} x$
   $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$; $\sec^{-1}(-x) = \pi - \sec^{-1} x$
7. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$, if $xy < 1$
   $= \pi + \tan^{-1} \frac{x+y}{1-xy}$ if $xy > 1$
   $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, \quad xy > -1$
   $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \frac{x+y+z-xyz}{1-xy-yz-zx}$
8. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}]$
   $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}]$
9. $2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2}$
   $2\sin^{-1} x = \sin^{-1}[2x\sqrt{1-x^2}]$; $2\cos^{-1} x = \cos^{-1}[2x^2 - 1]$
10. $3\tan^{-1} x = \tan^{-1} \frac{3x-x^3}{1-3x^2}$; $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$; $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$
11. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$; $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$; $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

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DOMAIN OF INVERSE CIRCULAR FUNCTION:

Table I

θ	Domain	Range
$\sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\text{cosec}^{-1} x$	$R-(-1,1)$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \theta \ne 0$
$\cos^{-1} x$	$-1 \le x \le 1$	$0 \le \theta \le \pi$
$\sec^{-1} x$	$R-(-1,1)$	$0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$
$\tan^{-1} x$	$-\infty < x < +\infty$ i.e., R	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$\cot^{-1} x$	$-\infty < x < +\infty$ i.e., R	$0 < \theta < \pi$

Example 1. Give the principal values of the following inverse functions :

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ and $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(iii) $\tan^{-1}(-\sqrt{3})$ and $\tan^{-1} 1$

Solution : (i) $\sin^{-1}(\sin\theta) = \theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $-1 \le x \le 1$

$\therefore$ $\sin^{-1}\frac{1}{2}$ is an angle lying in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and whose sine is $\frac{1}{2} \Rightarrow \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

Similarly $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ = an angle lying in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and whose sine is $-\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

(ii) $\because$ $\cos^{-1} x = \theta$, where $0 \le \theta \le \pi$ and $-1 \le x \le 1$

$\therefore$ $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ = an angle lying in $[0, \pi]$ whose value of cosine is $\frac{\sqrt{3}}{2}$

$\Rightarrow$ $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

Similarly $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ = an angle lying in $[0, \pi]$ whose cosine is $-\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

(iii) $\because$ $\tan^{-1} x = \theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ and whose tangent is $-\sqrt{3}$

$\therefore$ $\tan^{-1}(-\sqrt{3})$ = an angle lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is $-\sqrt{3}$

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$\Rightarrow \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

Similarly $\tan^{-1}(1)$ = an angle lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is 1

$\therefore \tan^{-1} 1 = \frac{\pi}{4}$

Example 2. Evaluate the following :

(i) $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$ (ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ (iii) $\tan^{-1}\left(\tan\frac{\pi}{6}\right)$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ (v) $\cos^{-1}\left(\frac{7\pi}{6}\right)$

Solution : (i) $\sin^{-1}(\sin\theta) = \theta$, if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Hence $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$ $\left[\because -\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$ if $0 \le \theta \le \pi$ $\therefore \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$ $\left[\because 0 \le \frac{2\pi}{3} \le \pi\right]$

(iii) $\because \tan^{-1}(\tan\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\therefore \tan^{-1}\left(\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$ $\left[\because -\frac{\pi}{2} < \frac{\pi}{6} < \frac{\pi}{2}\right]$

(iv) $\because \frac{2\pi}{3} \notin -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\therefore \sin^{-1}\left(\sin\frac{2\pi}{3}\right) \ne \frac{2\pi}{3}$

Now $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left\{\sin\left(\pi - \frac{\pi}{3}\right)\right\}$

$\Rightarrow \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \sin^{-1}\left(\sin\frac{\pi}{3}\right)$ $[\because \sin(\pi - \theta) = \sin\theta]$

$\Rightarrow \sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$

(v) $\because \frac{7\pi}{6} \notin [0, \pi]$ $\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) \ne \frac{7\pi}{6}$

Now $\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left\{\cos\left(2\pi - \frac{5\pi}{6}\right)\right\}$ $[\because \cos(2\pi - \theta) = \cos\theta]$

$\Rightarrow \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) \Rightarrow \cos^{-1}\cos\frac{7\pi}{6} = \frac{5\pi}{6}$ **Ans.**

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EXAMPLE: PROVE

(i) $2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\frac{\pi}{4}$ [U.P. 2018(SB)]

(ii) $\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$ [U.P. 2011, 18(O)]

Solution : (i) L.H.S. $=2\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}+\tan^{-1}\frac{1}{7}$

$$=\tan^{-1}\frac{\frac{2}{3}}{1-\frac{1}{9}}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{2}{3}}{\frac{8}{9}}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}=\tan^{-1}\frac{\frac{21+4}{28}}{\frac{28-3}{28}}=\tan^{-1}\frac{\frac{25}{28}}{\frac{25}{28}}$$

$$=\tan^{-1}1=\frac{\pi}{4}$$

Proved.

(ii) L.H.S. $=\tan^{-1}1+\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}1+\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$

$$=\tan^{-1}1+\tan^{-1}\frac{\frac{5}{6}}{\frac{5}{6}}=\tan^{-1}1+\tan^{-1}1=2\tan^{-1}1=2\times\frac{\pi}{4}=\frac{\pi}{2}$$

Proved.

EXAMPLE: SOLVE

$$4\sin^{-1}x+\cos^{-1}x=\pi$$

Solution : Given $4\sin^{-1}x+\cos^{-1}x=\pi$ or $3\sin^{-1}x+\sin^{-1}x+\cos^{-1}x=\pi$

or $3\sin^{-1}x+\frac{\pi}{2}=\pi$ $\quad\left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$

or $3\sin^{-1}x=\pi-\frac{\pi}{2}$ or $3\sin^{-1}x=\frac{\pi}{2}$

or $\sin^{-1}x=\frac{\pi}{6}$ $\therefore$ $x=\sin\frac{\pi}{6}=\frac{1}{2}$ **Ans.**