Unit II: ALGEBRA-II

Scalar Quantities and vector quantities:-

Scalar Quantities are defined as the physical quantities that have magnitude or size only. For example, distance, speed, mass, density, etc. However, **Vector quantities** are those physical quantities that have both magnitude and direction like displacement, velocity, acceleration, force, mass, etc.

We have listed the various differences between a scalar and vector in the table below:

	Vector	**Scalar**
Definition	A physical quantity with both the magnitude and direction.	A physical quantity with only magnitude.
Representation	A number (magnitude), direction using unit cap or arrow at the top and unit.	A number (magnitude) and unit
Symbol	Quantity symbol in bold and an arrow sign above	Quantity symbol
Direction	Yes	No
Example	Velocity and Acceleration	Mass and Temperature

Vector Notation

For vector quantity usually, an arrow is used on the top as shown below, which represents the vector value of the velocity and also explains that the quantity has both magnitudes as well as direction.

$$VectorNotation = \vec{v}$$

Types of Vectors

There are 10 types of vectors in mathematics which are:

1. Zero Vector
2. Unit Vector
3. Position Vector
4. Co-initial Vector
5. Like and Unlike Vectors

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6. Co-planar Vector
7. Collinear Vector
8. Equal Vector
9. Displacement Vector
10. Negative of a Vector

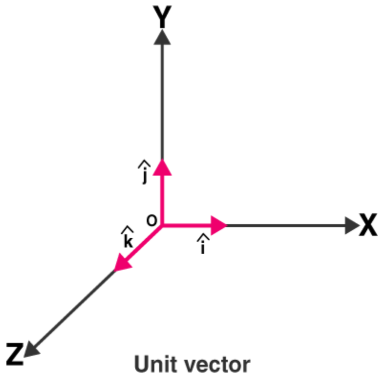
Zero Vector

A zero vector is a vector when the magnitude of the vector is zero and the starting point of the vector coincides with the terminal point.

Unit Vector

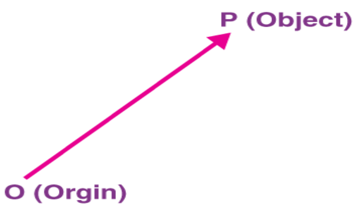
A vector which has a magnitude of unit length is called a unit vector.

Suppose if $\vec{x}$ is a vector having a magnitude x then the unit vector is denoted by $\hat{x}$ in the direction of the vector $\vec{x}$ and has the magnitude equal to 1.



Position Vector

If O is taken as reference origin and P is an arbitrary point in space then the vector $\vec{OP}$ is called as the position vector of the point.



Co-initial Vectors

The vectors which have the same starting point are called co-initial vectors.

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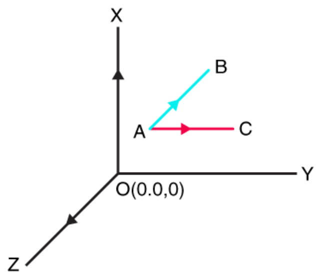


The vectors $\vec{AB}$ and $\vec{AC}$ are called co − initial vectors as they have the same starting point.

Like and Unlike Vectors

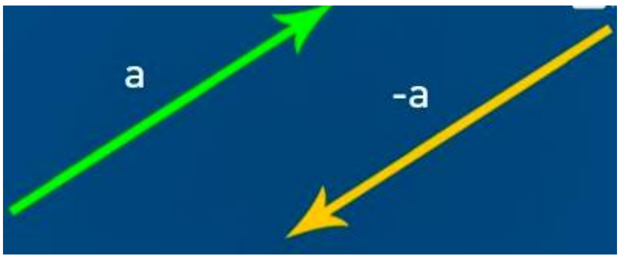
The vectors having the same direction are known as like vectors. On the contrary, the vectors having the opposite direction with respect to each other are termed to be unlike vectors.

Co-planar Vectors

Three or more vectors lying in the same plane or parallel to the same plane are known as co-planar vectors.

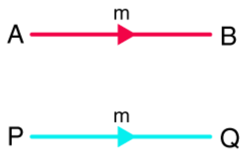
Collinear Vectors

Vectors that lie along the same line or parallel lines are known to be collinear vectors. They are also known as parallel vectors.



Equal Vectors

Two or more vectors are said to be equal when their magnitude is equal and also their direction is the same.



The two vectors shown above, are equal vectors as they have both direction and magnitude equal.

Displacement Vector

If a point is displaced from position A to B then the displacement AB represents a vector $\overrightarrow{AB}$ which is known as the displacement vector.

Negative of a Vector

If two vectors are the same in magnitude but exactly opposite in direction then both the vectors are negative of each other. Assume there are two vectors **a** and **b,** such that these vectors are exactly the same in magnitude but opposite in direction then these vectors can be given by

a = – **b**

Vector Addition and Subtraction :-

The addition and subtraction of vector quantities do not follow the simple arithmetic rules. A special set of rules are followed for the addition and subtraction of vectors. Following are some points to be noted while adding vectors:

- Addition of vectors means finding the resultant of a number of vectors acting on a body.
- The component vectors whose resultant is to be calculated are independent of each other. Each vector acts as if the other vectors were absent.
- Vectors can be added geometrically but not algebraically.
- Vector addition is commutative in nature, i.e.,

$$\vec{A}+\vec{B}=\vec{B}+\vec{A}$$

Now, about vector subtraction, it is the same as adding the negative of the vector to be subtracted.

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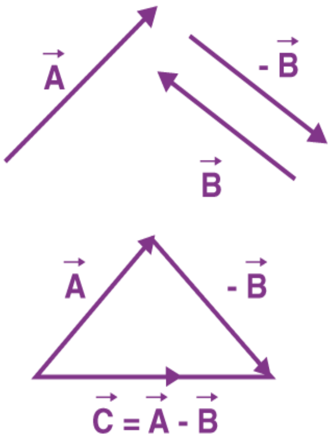
To better understand, let us look at the example given below.

Let us consider two vectors, A and B, as shown in the figure below. We need to subtract vector B from vector A. It is just the same as adding vector B and vector A. The resultant vector is shown in the figure below.



Q. What is the magnitude of a unit vector?

Answer: The magnitude of a unit vector is unity. A unit vector has no units or dimensions.

Algebraic Properties of Vectors

- Commutative (vector) P + Q = Q + P.
- Associative (vector) (P + Q) + R = P + (Q + R)
- Additive identity There is a vector 0 such. ...
- Additive inverse For any P there is a vector -P such that P + (-P) = 0.
- Distributive (vector) r(P + Q) = rP + rQ.
- Distributive (scalar) (r + s) P = rP + sP.

components of a vector :-

The **components of a vector** in two dimension coordinate system are usually considered to be x-component and y-component. It can be represented as, $V = (V_x, V_y)$, where V is the vector. These are the parts of vectors generated along the axes.

Where V is the magnitude of vector V and can be found using Pythagoras theorem;

$$|V| = \sqrt{(v_x^2, v_y^2)}$$

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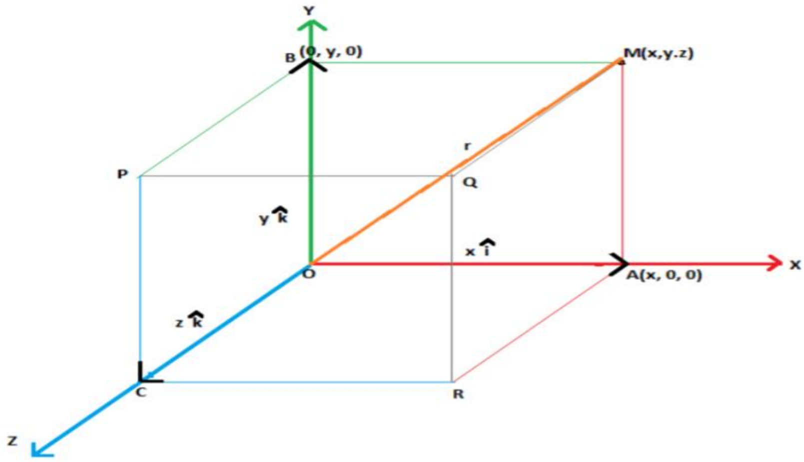
Components of vector in 3D

To represent a vector in space, we resolve the vector along the three mutually perpendicular axes as shown below.

The vector OM can be resolved along the three axes as shown. With OM as the diagonal, a parallelepiped is constructed whose edges OA, OB and OC lie along the three perpendicular axes.



The vector can be represented as

r = $\vec{OM} = x\hat{i} + y\hat{j} + z\hat{k}$

This is known as the component form of a vector.

$$\Rightarrow |\boldsymbol{r}| = \sqrt{(\boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2})}$$

Example: Two vectors are given by a = $5\hat{i} - 3\hat{j} + 4\hat{k}$ and b = $2\hat{i} - \hat{j} + \hat{k}$. Find the unit vectors and the sum and difference of both the vectors.

Solution: The unit vector is given by

$\hat{x} = \frac{\vec{x}}{|\vec{x}|}$

The magnitude of both the vectors can be given as:

|a| = $\sqrt{5^2 + (-3)^2 + 4^2}$

⇒ |a| = $\sqrt{50}$

|b| = $\sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

Now, the unit vectors can be given as:

$\hat{a} = \frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{50}}$

$$\hat{b} = \frac{2\hat{i} - \hat{j} + \hat{k}}{\sqrt{6}}$$

The sum can be given by:
a + b = $5\hat{i} - 3\hat{j} + 4\hat{k} + 2\hat{i} - \hat{j} + \hat{k}$
The difference is given by:
a – b = $3\hat{i} - 2\hat{j} + 3\hat{k}$

Direction Cosine :-

The direction cosine of the vector can be determined by dividing the corresponding coordinate of a vector by the vector length. The unit vector coordinate is equal to the direction cosine. One such property of the direction cosine is that the addition of the squares of the direction cosines is equivalent to one.

We know that the direction cosine is the cosine of the angle subtended by the line with the three coordinate axes, such as x-axis, y-axis and z-axis respectively. If the angles subtended by these three axes are α, β, and γ, then the direction cosines are cos α, cos β, cos γ respectively. The direction cosines are also represented by l, m, and n.

Thus, the direction cosine of a vector $\vec{A} = a\hat{i} + b\hat{j} + c\hat{k}$ is given as:

$$cos\alpha = l = \frac{a}{\sqrt{(a)^2 + (b)^2 + (c)^2}}$$
$$cos\beta = m = \frac{b}{\sqrt{(a)^2 + (b)^2 + (c)^2}}$$
$$cos\gamma = n = \frac{c}{\sqrt{(a)^2 + (b)^2 + (c)^2}}$$

Example : Determine the direction cosine of a line joining the point (-4, 2, 3) with the origin.

Solution: Given that, the line joins the origin (0, 0, 0) and the point (-4, 2, 3). Hence, the direction ratios are -4, 2, 3.
Also, the magnitude of a line = $\sqrt{[(-4)^2+(2)^2+(3)^2]}$
= √(16+4+9) = √29.
Therefore, the direction cosines are ((-4/√29), (2/√29), (3/√29)).

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Example : Find the direction cosine of a vector joining the points A(1, 2, -3) and B(-1, -2, 1), directed from A to B.

Solution:

Given that, A(1, 2, -3) and B(-1, -2, 1)

$$\vec{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + (1-(-3))\hat{k}$$

$$\vec{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$$

Hence, the direction ratios are -2, -4, 4.

Magnitude = $\sqrt{[(-2)^2+(-4)^2+(4)^2]}$

= $\sqrt{(4+16+16)} = \sqrt{36} = 6$.

Thus, the direction cosines are (-2/6, -4/6, 4/6), which is also equal to (-⅓, -⅔, ⅔)

Definition. The vectors $a_1, \ldots, a_n$ are called linearly independent if there are no non-trivial combination of these vectors equal to the zero vector.

That is, the vector $a_1, \ldots, a_n$ are linearly independent if $x_1 a_1 + \ldots + x_n a_n = 0$ if and only if $x_1 = 0, \ldots, x_n = 0$.

Definition. The vectors $a_1, \ldots, a_n$ are called linearly dependent if there exists a non-trivial combination of these vectors is equal to the zero vector.

Q.

Determine if the following set of vectors is linearly independent:

$$\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$$

I've done the following system of equations, and I think I did it right... It's been such a long time since I did this sort of thing...

Assume the following:

$$a\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Determine if $a = b = c = 0$:

$$\begin{aligned} 2a + b + 4c &= 0 \qquad (1) \\ 2a - b + 2c &= 0 \qquad (2) \\ b - 2c &= 0 \qquad (3) \end{aligned}$$

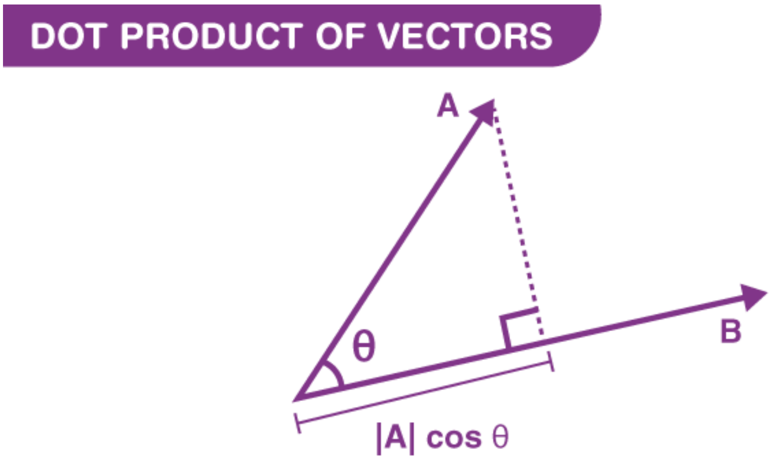
Subtract (2) from (1):

$$\begin{aligned} b + c &= 0 \qquad (4) \\ b - 2c &= 0 \qquad (5) \end{aligned}$$

Substitute (5) into (4), we get $c = 0$.

When c=0
We can say that according to the condition
a=b=c=0
So when c=0 then
a=0 , b=0.
So, the vectors are linearly independent .

Dot Product of two Vectors :-

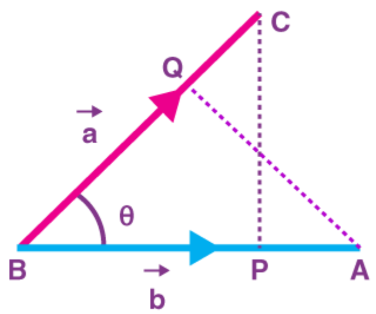


The scalar product of two vectors **a** and **b** of magnitude **|a|** and **|b|** is given as **|a||b|** cos θ, where θ represents the angle between the vectors **a** and **b** taken in the direction of the vectors.

We can express the scalar product as:

a.b=|a||b| cosθ

Projection of Vectors



BP is known to be the projection of a vector **a** on vector **b** in the direction of vector **b** given by **|a|** cos θ.
Similarly, the projection of vector **b** on a vector **a** in the direction of the vector **a** is given by **|b|** cos θ.
Projection of vector **a** in direction of vector **b** is expressed as

$$\Rightarrow \vec{BP} = \frac{a.b}{|b|^2} b$$

Similarly, projection of vector **b** in direction of vector **a** is expressed as

$$\Rightarrow \vec{BQ} = \frac{a.b}{|a|^2} a$$

Dot Product Properties of Vector:

- **Property 1:** Dot product of two vectors is commutative i.e. **a.b = b.a** = ab cos θ.
- **Property 2:** If **a.b = 0** then it can be clearly seen that either **b** or **a** is zero or cos θ = 0.

  $\Rightarrow \theta = \frac{\pi}{2}$..

  It suggests that either of the vectors is zero or they are perpendicular to each other.

- **Property 3:** Also we know that using scalar product of vectors (p**a**).(q**b**)=(p**b**).(q**a**)=pq **a.b**
- **Property 4:** The dot product of a vector to itself is the magnitude squared of the vector i.e. a.a = a.a cos 0 = a^2
- **Property 5:** The dot product follows the distributive law also i.e. **a.(b + c) = a.b + a.c**
- **Property 6:** In terms of orthogonal coordinates for mutually perpendicular vectors it is seen that

$$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

- **Property 7:** In terms of unit vectors, if

$$a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Then

$$a.b = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = ab\cos\theta$$

Example 1: Let there be two vectors [6, 2, -1] and [5, -8, 2]. Find the dot product of the vectors.

Solution:

Given vectors: [6, 2, -1] and [5, -8, 2] be a and b respectively.

a.b = (6)(5) + (2)(-8) + (-1)(2)

a.b = 30 – 16 – 2

a.b = 12

Example 2: Let there be two vectors |a|=4 and |b|=2 and θ = 60°. Find their dot product.

Solution:

a.b = |a||b|cos θ

a.b = 4.2 cos 60°

a.b = 4.2 × (1/2)

a.b = 4

EXAMPLE 1 Find the Dot Product to Determine Orthogonal Vectors in Space

A. Find the dot product of u and v for u = ⟨–1, 6, –3⟩ and v = ⟨3, –1, –3⟩. Then determine if u and v are orthogonal.

u • v = –1(3) + 6(–1) + (–3)(–3)

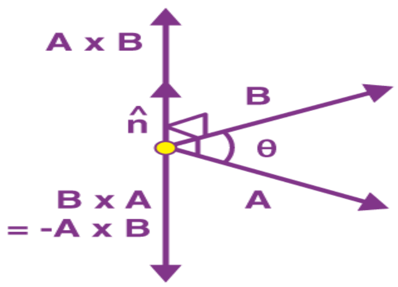
= –3 + (–6) + 9 or 0

Since **u • v** = 0, **u** and **v** are orthogonal.

Answer: 0; orthogonal

Vector product (Cross Product) of two vectors :-

The Vector product of two vectors, **a** and **b**, is denoted by **a** × **b**. Its resultant vector is perpendicular to **a** and **b**. Vector products are also called cross products. Cross product of two vectors will give the resultant a vector and calculated using the Right-hand Rule.



Cross Product Formula :-

A × **B** =|**A**| |**B**| sin θ Or,

$$\vec{A} \times \vec{B} = ||\vec{A}||\ ||\vec{B}||sin\theta\ \hat{n}$$

Here,

$$\vec{A}, \vec{B}\ are\ the\ two\ vectors.$$

$$\left|\left|\vec{A}\right|\right|,\ \left|\left|\vec{B}\right|\right|\ are\ the\ magnitudes\ of\ given\ vectors.$$

θ is the angle between two vectors

$\hat{n}$ is the unit vector perpendicular to the plane containing the given two vectors, in the direction given by the right – hand rule.

NOTE : i × j = k and j × i = -k j × k = i and k × j = -i k × i = j and i × k = -j

Also, the anti-commutativity of the cross product and the distinct absence of linear independence of these vectors signifies that:

i × i = j × j = k × k = 0

Cross Product Matrix :-

We can also derive the formula for the cross product of two vectors using the determinant of the matrix as given below.

A = ai + bj + ck B = xi + yj + zk

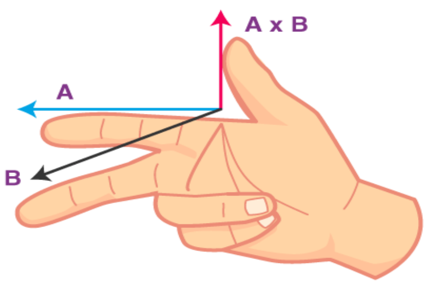
Thus,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix}$$

A × B = (bz – cy)i – (az – cx)j + (ay – bx)k = (bz – cy)i + (cx – az)j + (ay – bx)k

Right-hand Rule Cross Product :-

We can find the direction of the unit vector with the help of the right-hand rule.



Cross Product Properties :-

1. Anti-commutative Property : $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
2. Distributive Property : $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
3. Jacobi Property: $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$
4. Zero Vector Property: $a \times b = 0$ if a = 0 or b =0.

Cross Product of Perpendicular Vectors :-

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}| sin\theta$$

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}| sin 90^\circ$$

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}|,$$

which is equal to the area of a rectangle.

Hence, the cross product of the perpendicular vectors becomes

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}|$$

Cross Product of Parallel vectors :-

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}|sin\theta$$

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}|sin0^\circ$$

$$\vec{X} \times \vec{Y} = |\vec{X}|.\,|\vec{Y}| \times 0$$

Hence, the cross product of the parallel vectors becomes

$\vec{X} \times \vec{Y} = 0$, which is a unit vector.

Magnitude of Cross Product :-

Let us assume two vectors,

$\vec{A} = A_x + A_y + A_z$

$$\vec{B} = B_x + B_y + B_z$$

Then the magnitude of two vectors is given by the formula,

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Hence, the magnitude of the cross product of two vectors is given by the formula,

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}||sin\theta|$$

Example: Find the cross product of the given two vectors:

$$\vec{X} = 5\vec{i} + 6\vec{j} + 2\vec{k} \text{ and } \vec{Y} = \vec{i} + \vec{j} + \vec{k}$$

Solution: Given:

$\vec{X} = 5\vec{i} + 6\vec{j} + 2\vec{k}$ and $\vec{Y} = \vec{i} + \vec{j} + \vec{k}$

To find the cross product of two vectors, we have to write the given vectors in determinant form. Using the determinant form, we can find the cross product of two vectors as:

$$\vec{X} \times \vec{Y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 6 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

By expanding, $\vec{X} \times \vec{Y} = (6-2)\vec{i} - (5-2)\vec{j} + (5-6)\vec{k}$

Therefore, $\vec{X} \times \vec{Y} = 4\vec{i} - 3\vec{j} - \vec{k}$

Q.

$$\text{Find } \vec{a} \times \vec{b} \text{ if } \vec{a} = 2\hat{i} + \hat{k} \text{ and } \vec{b} = \hat{i} + \hat{k} + \hat{k}.$$

Solution: Given

$\vec{a} = 2\hat{i} + 0\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{k} + \hat{k}$

So

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

= i(0 – 1) – j(2 – 1) + k(2 – 0)

$$= -\hat{i} - \hat{j} + 2\hat{k}$$

Scalar Triple Product:-

If it is given that

$$a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$c = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Then, it is evident that the scalar triple product of vectors means the product of three vectors. It means taking the dot product of one of the vectors with the cross product of the remaining two. It is denoted as

$$[a\ b\ c] = (a \times b) \cdot c$$

The following conclusions can be drawn, by looking into the above formula:

i) The resultant is always a scalar quantity.

ii) Cross product of the vectors is calculated first, followed by the dot product which gives the scalar triple product.

iii) The physical significance of the scalar triple product formula represents the volume of the parallelepiped whose three coterminous edges represent the three vectors a, b and c. The following figure will make this point more clear.

we can conclude that for a Parallelepiped, if the coterminous edges are denoted by three vectors and **a, b** and **c** then,

Volume of parallelepiped = (a × b) **c cos α =** (a × b) . c

Where α is the angle between **(a × b)** and **c.**

then, we can express the above equation as,

$$(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}.(\boldsymbol{c_1}\hat{\boldsymbol{i}} + \boldsymbol{c_2}\hat{\boldsymbol{j}} + \boldsymbol{c_3}\hat{\boldsymbol{k}})$$

$$(\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = \begin{vmatrix} \hat{\boldsymbol{i}}.(\boldsymbol{c_1}\hat{\boldsymbol{i}} + \boldsymbol{c_2}\hat{\boldsymbol{j}} + \boldsymbol{c_3}\hat{\boldsymbol{k}}) & \hat{\boldsymbol{j}}.(\boldsymbol{c_1}\hat{\boldsymbol{i}} + \boldsymbol{c_2}\hat{\boldsymbol{j}} + \boldsymbol{c_3}\hat{\boldsymbol{k}}) & \hat{\boldsymbol{k}}.(\boldsymbol{c_1}\hat{\boldsymbol{i}} + \boldsymbol{c_2}\hat{\boldsymbol{j}} + \boldsymbol{c_3}\hat{\boldsymbol{k}}) \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_2} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}$$

$$\hat{\boldsymbol{i}}.\hat{\boldsymbol{i}} = \hat{\boldsymbol{j}}.\hat{\boldsymbol{j}} = \hat{\boldsymbol{k}}.\hat{\boldsymbol{k}} = \mathbf{1}\ (\mathbf{As\ cos\ 0\ =\ 1})$$

$$\Rightarrow \hat{\boldsymbol{i}}.(\boldsymbol{c_1}\hat{\boldsymbol{i}} + \boldsymbol{c_2}\hat{\boldsymbol{j}} + \boldsymbol{c_3}\hat{\boldsymbol{k}}) = \boldsymbol{c_1}$$

$$\Rightarrow \hat{\boldsymbol{j}}.(\boldsymbol{c_1}\hat{\boldsymbol{i}} + \boldsymbol{c_2}\hat{\boldsymbol{j}} + \boldsymbol{c_3}\hat{\boldsymbol{k}}) = \boldsymbol{c_2}$$

$$\Rightarrow \hat{k}.(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = c_3$$

$$\Rightarrow (\boldsymbol{a} \times \boldsymbol{b}).\boldsymbol{c} = \begin{vmatrix} \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_3} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}$$

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$$[\boldsymbol{a}\ \boldsymbol{b}\ \boldsymbol{c}] = \begin{vmatrix} \boldsymbol{c_1} & \boldsymbol{c_2} & \boldsymbol{c_3} \\ \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \\ \boldsymbol{b_1} & \boldsymbol{b_2} & \boldsymbol{b_3} \end{vmatrix}$$

Properties of Scalar Triple Product:

i) If the vectors are cyclically permuted, then **(a × b) . c = a.(b × c)**

ii) The product is cyclic in nature, i.e., a.(b × c) = b.(c × a) = c.(a × b)

Thus,

[a b c] = [b c a] = [c a b] = – [b a c] = – [c b a] = – [a c b]

iii) [a a b] = [a b b] = [a c c] = 0

Example: Three vectors are given by,

$$\boldsymbol{a} = \hat{\boldsymbol{i}} - \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

$$\boldsymbol{b} = 2\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$$

$$\boldsymbol{c} = \hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}}$$

By using the scalar triple product of vectors, verify that

[a b c] = [b c a] = – [a c b]

Solution: First of all let us find **[a b c]**.

[a b c] = (a × b) . c

$$\text{We know that } [a\ b\ c] = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\Rightarrow [a\ b\ c] = \begin{vmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

⇒ **[a b c]** = 1(-1 – 1) – 1(1 – 2) – 2(1 + 2) = -2 + 1 – 6 = -7

Now let us evaluate **[b c a]** and **[a c b]** similarly,

$$\Rightarrow [b\ c\ a] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

= 1(-2 – 1) + 1(-4 – 1) + 1(2 – 1) = -3 – 5 + 1 = -7

$$\Rightarrow [a\ c\ b] = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & a_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$$

= 2(2 – 1) – 1(-2 – 1) + 1(1 + 1) = 2 + 3 + 2 = 7

Hence it can be seen that **[a b c] = [b c a] = – [a c b]**

Unit II: ALGEBRA-II

Triple Cross Product

The **product of three vectors** is called the triple product; in other words, the cross product of one vector with the cross product of another two vectors.
If we have three vectors, $\vec{A}, \vec{B},$ and $\vec{C},$
then the vector triple product is denoted as follows:

A × (B × C) = (A . C) B − (A . B) C

(A × B) × C = −C × (A × B) = −(C . B) A + (C . A) B

Cross Product Rules:-

Anti-Commutative Property

The cross product is an anti-commutative property
It means $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

Distributive Property:-

The cross product has the distributive property over addition.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Jacobi Property:-

Cross-product will satisfy the Jacobi property.

$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$

Zero Vector Property:-

$$a \times b = 0 \; if \; a = 0 \; or \; b = 0$$

Important results :-

1. The area of a parallelogram with adjacent sides is $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$
2. The area of a triangle whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$
3. The area of a triangle ABC is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$ or $\frac{1}{2}|\vec{BC} \times \vec{BA}|$ or $\frac{1}{2}|\vec{CB} \times \vec{CA}|$
4. The area of a parallelogram with diagonals is

$\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$

5. The area of a plane quadrilateral ABCD is $\frac{1}{2}\left|\vec{AC} \times \vec{BD}\right|$, AC and BD are the diagonals of it.

Cross Product in Spherical Coordinates :-

The resultant vector of the cross product of two vectors is perpendicular to both vectors, and it is normal to the plane in which they lie. We can use spherical coordinates in a 3-dimensional system to represent the same. We can define any vector in a 3-dimension system as, first, radical distance r, i.e. the distance of a fixed point to the origin, and second, the polar angle θ and third, azimuth angle ϕ.

We know that the transformation from Cartesian to spherical is

X = r sin θ cos ϕ

Y = r sin θ sin ϕ

Z = r cos θ

Example : A = i + 2j + 3k, B = 2i + 3j − 2k, C = i + j + k.

Find A × (B × C), if A × (B × C) = (A . C) B − (A . B) C

Solution:

A.C = (i + 2j + 3k) . (i + j + k) = 6

A.B = (i + 2j + 3k) . (2i + 3j − 2k) = 2

Then, A × (B × C) = 6(2i + 3j − 2k) − 2(i + j + k)

= 10i − 16j − 14k

Example : Find the unit vector to the two vectors:

$$\vec{A} = 2\hat{i} + (-1)\hat{j} + 3\hat{k},\ \vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$$

Solution: Let $\vec{C}$ be the unit vector.

Cross-product of A and B,

$$\vec{A} \times \vec{B} = 4\hat{i} - \hat{j} - 3\hat{k}$$

Now unit vector in the direction of $\vec{C}$ is $\frac{\vec{C}}{|\vec{C}|}$.

$$|\vec{C}| = \sqrt{26}$$

Therefore desired unit vector is (1/√26) (4i – j – 3k)

Example : What is the number of vectors of unit length perpendicular to vectors a = (1, 1, 0) and b = (0, 1, 1)?

Solution:

The vector perpendicular to **a** and **b** is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Since the length of this vector is √3, the unit vector perpendicular to **a** and **b** is:

$$\pm \frac{a \times b}{|a \times b|}$$

$$= \pm \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Hence, there are two such vectors.

IMAGINARY NUMBER:-

An imaginary number is a real number multiplied by the imaginary unit i, which is defined by its property $\mathbf{i^2 = -1}$. The square of an imaginary number bi is $-b^2$.

For example, 5i is an imaginary number, and its square is −25.

the rules for some imaginary numbers are:

- $i = \sqrt{-1}$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = +1$
- $i^{4n} = 1$
- $i^{4n-1} = -i$

Example: Solve the imaginary number i^7

Solution: The given imaginary number is i^7

Now, split the imaginary number into terms, and it becomes

$i^7 = i^2 \times i^2 \times i^2 \times i$

$i^7 = -1 \times -1 \times -1 \times i$

$i^7 = -1 \times i$

$i^7 = – i$

Therefore, i^7 is – i.

Complex number :-

Complex numbers are the combination of both real numbers and imaginary numbers. The complex number is of the standard form: a + bi

Where

a and b are real numbers

i is an imaginary unit.

Real Numbers Examples : 3, 8, -2, 0, 10

Imaginary Number Examples: 3i, 7i, -2i, √i

Complex Numbers Examples: 3 + 4 i, 7 – 13.6 i, 0 + 25 i = 25 i, 2 + i.

Addition of Numbers Having Imaginary Numbers

When two numbers, a+bi, and c+di are added, then the real parts are separately added and simplified, and then imaginary parts separately added and simplified. Here, the answer is (a+c) + i(b+d).

Subtraction of Numbers Having Imaginary Numbers

When c+di is subtracted from a+bi, the answer is done like in addition. It means, grouping all the real terms separately and imaginary terms separately and doing simplification. Here, (a+bi)-(c+di) = (a-c) +i(b-d).

Multiplication of Numbers Having Imaginary Numbers

Consider (a+bi)(c+di)

It becomes:

(a+bi)(c+di) = (a+bi)c + (a+bi)di

$= ac+bci+adi+bdi^2$

= (ac-bd)+i(bc+ad)

Division of Numbers Having Imaginary Numbers

Consider the division of one imaginary number by another.

(a+bi) / (c+di)

Multiply both the numerator and denominator by its conjugate pair, and make it real. So, it becomes

$(a+bi) / (c+di) = (a+bi) (c-di) / (c+di) (c-di) = [(ac+bd)+ i(bc-ad)] / c^2 +d^2$.

Equality of Complex Numbers

Assume that z_1 and z_2 are the two complex numbers.

Here $z_1 = a_1+i\, b_1$ and $z_2 = a_2+ib_2$

If both the complex numbers z_1 and z_2 are equal (i.e) $z_1 = z_2$, then we can say that the real part of the first complex number is equal to the real part of the second complex number, whereas the imaginary part of the first complex number is equal to the imaginary part of the second complex number.

(i.e) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(Z_2)$

Thus, the equality of complex number states that,

if $a_1+ib_1 = a_2+ib_2$, then $a_1 = a_2$ and $b_1 = b_2$.

Example: $z_1 = a+3i$, $z_2 = 4+bi$, $z_3 = 6+10i$.Find the value of a and b if $z_3 = z_1+z_2$

Solution:

By the definition of addition of two complex numbers,

$Re(z_3) = Re(z_1)+Re(z_2)$

$6 = a + 4$

$a = 6 – 4 = 2$

$Im(z_3) = Im(z_1) + Im(z_2)$

$10 = 3+b$

$b = 10-3 =7$

Example: $z_1 =4+ai, z_2=2+4i, z_3 =2$. Find the value of a if $z_3=z_1-z_2$

Solution:

By the definition of difference of two complex numbers,

$Im_3=Im_1-Im_2$

$0 = a – 4$

a = 4

Example: z_1=6-2i, z_2=4+3i. Find z_1 z_2

Solution:

z_1 z_2 = (6-2i) (4+3i)

= 6 × 4 + 6 × 3i + (-2i) × 4 + (-2i)(3i)

= 24 + 18i – 8i – 6i^2

= 24 + 10i + 6

= 30 + 10i

Multiplicative inverse of a complex number

Definition: For any non-zero complex number z=a+ib(a≠0 and b≠0) there exists another complex number z^{-1} or 1/z, which is known as the multiplicative inverse of z such that zz^{-1} = 1.

z = a+ib, then,

$$z^{-1} = \frac{a}{a^2 + b^2} + i\frac{(-b)}{a^2 + b^2}$$
$$Re(z^{-1}) = \frac{a}{a^2 + b^2}$$
$$Im(z^{-1}) = \frac{-b}{a^2 + b^2}$$

Example: z = 3 + 4i. Find the inverse of z.

Solution: $z^{-1}\ of\ a + ib = \frac{a}{a^2+b^2} + i\frac{(-b)}{a^2+b^2} = \frac{(a-ib)}{a^2+b^2}$

The numerator of z^{-1} is conjugate of z, that is a – ib

Denominator of z^{-1} is sum of squares of the Real part and imaginary part of z

Here, z = 3 + 4i

$$z^{-1} = \frac{3 - 4i}{3^2 + 4^2} = \frac{3 - 4i}{25}$$
$$z^{-1} = \frac{3}{25} - \frac{4i}{25}$$

Example: If $z_1 = 2 + 3i$ and $z_2 = 1 + i$, find z_1/z_2.

Solution: $\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2}$

$$\frac{2+3i}{1+i} = (2+3i) \times \frac{1}{1+i}$$

$$Since, \frac{1}{1+i} = \frac{1-i}{1^2+1^2} = \frac{1-i}{2}$$

$$\frac{2+3i}{1+i} = 2+3i \times \frac{1-i}{2} = \frac{(2+3i)(1-i)}{2}$$

$$= \frac{2-2i+3i-3i^2}{2} = \frac{5+i}{2}$$

Conjugate of Complex number

Conjugate of a complex number z=a+ib is given by changing the sign of the imaginary part of z which is denoted as

$$\bar{z}$$

$$\bar{z} = a - ib$$
$$z + \bar{z} = 2a$$
$$z - \bar{z} = 2bi$$

Properties of the Conjugate of a Complex Number

Below are some properties of the conjugate of complex numbers, along with their proof

(1) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(2)

$\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$

Proof: let z_1= a + ib and z_2= c + id

Then, $\overline{z_1.z_2} = \overline{(a+ib).(c+id)}$

$\Rightarrow \overline{ac + iad + ibc + i^2bd}$

$\Rightarrow \overline{ac + iad + ibc - bd}$

$\Rightarrow \overline{ac - bd + i(ad + bc)}$

$\Rightarrow ac - bd - i(ad + bc)$

$\Rightarrow ac + i^2bd - iad - ibc$

$\Rightarrow ac - ibc + i^2bd - iad$

$\Rightarrow c(a - ib) - id(a - ib)$

$\Rightarrow (a - ib).(c - id)$

$\Rightarrow \overline{z_1}.\overline{z_2}$

3. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Proof: $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(z_1.\frac{1}{z_2}\right)}$

Using multiplicative property of conjugate we have

$\Rightarrow \overline{z_1}.\overline{\left(\frac{1}{z_2}\right)}$

$\Rightarrow \frac{\overline{z_1}}{\overline{z_2}}$

4. $\overline{\overline{z}}$ = z

Proof: let z = a + ib

Then, $\overline{\overline{z}} = \overline{\overline{a+ib}} = \overline{a-ib}$ = a + ib = z.

5. If z = a + ib

Then, z.$\overline{z} = a^2 + b^2 = |z|^2$

Proof: z.$\overline{z} = (a+ib)\,\overline{(a+ib)}$ = (a + ib)(a - ib) = $a^2 - i^2b^2 = a^2 + b^2 = |z|^2$

Points to Remember:

- $z + \bar{z} = 2\ Re(z)$
- $z - \bar{z} = 2i\ Im(z)$
- If z lies in the 1st quadrant, then

$\bar{z}$ will lie in the 4th quadrant, and $-\bar{z}$ will lie in the 2nd quadrant.

- If x + iy = f(a + ib) then x – iy = f(a – ib)
- Further, g(x + iy) = f(a + ib) ⇒ g(x – iy) = f(a – ib).

Modulus of a Complex Number :-

The modulus of the complex number is the distance of the point on the Argand plane representing the complex number z from the origin. Let P is the point that denotes the complex number, z = x + iy. Then, OP **= |z| = √(x^2 + y^2).**

Note:

1. |z| > 0.

2. All the complex numbers with the same modulus lie on the circle with the centre origin and radius r = |z|.

Properties of Modulus of Complex Number

Below are a few important properties of the modulus of a complex number and their proofs.

(i) $|z_1 z_2| = |z_1||z_2|$

(ii) $|z_1 / z_2| = (|z_1|) / (|z_2|)$.

Some other important results:

Triangle inequalities:

$|z_1 + z_2| \le |z_1| + |z_2|$

$|z_1 + z_2| \ge |z_1| - |z_2|$

$|z_1 - z_2| \ge |z_1| - |z_2|$.

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Example 1: Find the conjugate of the complex number z = (1 + 2i)/(1 – 2i).

Solution: z = (1 + 2i)/(1 – 2i)

Rationalising given the complex number, we have

⇒ z = ((1 + 2i)/(1 – 2i))× (1 + 2i)/(1 + 2i)

⇒ $z = (1 + 2i)^2/(1^2 - (2i)^2)$

⇒ $z = (1 + 4i^2 + 4i)/(1 + 4)$

⇒ z = (1 – 4 + 4i)/(1 + 4)

⇒ z = (-3 + 4i)/5

⇒

$$\Rightarrow \bar{z} = \frac{(-3- 4i)}{5}$$

Example 2: Find the modulus of the complex number z = (3 – 2i)/2i

Solution: z = (3 – 2i)/2i

⇒ z = (3)/2i – 2i/2i

⇒ z = 3/2i – 1

⇒ $z = 3i/(2i^2) - 1$

⇒ z = (-3i/2) – 1

⇒ $|z| = \sqrt{(-\frac{3}{2})^2 + (-1)^2}$

⇒ $|z| = \sqrt{\frac{9}{4} + 1}$

⇒ $|z| = \sqrt{\frac{9+4}{4}}$

⇒ $|z| = \sqrt{\frac{13}{4}}$

⇒ $|z| = \frac{\sqrt{13}}{2}$

Example 3: If z + |z| = 1 + 4i, then find the value of |z|.

Solution: Let z = x + iy

⇒ z + |z| = 1 + 4i

⇒ x + iy + |x + iy| = 1 + 4i

⇒ $x + iy + \sqrt{(x^2 + y^2)} = 1 + 4i$

⇒ y = 4 and $x + \sqrt{(x^2 + y^2)} = 1$

$\Rightarrow x + \sqrt{(x^2 + 4^2)} = 1$

$\Rightarrow \sqrt{(x^2 + 4^2)} = 1 - x$

Squaring both sides, we have

$\Rightarrow x^2 + 4^2 = 1 + x^2 - 2x$

$\Rightarrow 2x = -15$

or $x = -15/2$

$$\Rightarrow |z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow |z| = \sqrt{(-\frac{15}{2})^2 + 4^2}$$

$$\Rightarrow |z| = \sqrt{\frac{225}{4} + 16}$$

$$\Rightarrow |z| = \sqrt{\frac{225 + 64}{4}}$$

$$\Rightarrow |z| = \sqrt{\frac{289}{4}}$$

$$\Rightarrow |z| = \frac{17}{2}$$

Polar Form Equation

The equation of polar form of a complex number $z = x+iy$ is:

$z=r(\cos\theta+i\sin\theta)$

where

$r=|z|=\sqrt{(x^2+y^2)}$

$x=r \cos\theta$

$y=r \sin\theta$

$\theta=\tan^{-1}(y/x)$ for $x>0$

$\theta=\tan^{-1}(y/x)+\pi$ or

$\theta=\tan^{-1}(y/x)+180°$ for $x<0$.

Converting Rectangular form into Polar form

Let us see some examples of conversion of the rectangular form of complex numbers into polar form.

Unit II: ALGEBRA-II

Example: Find the polar form of complex number 7-5i.

Solution:7-5i is the rectangular form of a complex number.

To convert into polar form modulus and argument of the given complex number, i.e. r and θ.

We know, the modulus or absolute value of the complex number is given by:

$r=|z|=\sqrt{x^2+y^2}$

$r=\sqrt{(7^2+(-5)^2}$

$r=\sqrt{49+25}$

$r=\sqrt{74}$

r=8.6

To find the argument of a complex number, we need to check the condition first, such as:

Here x>0, therefore, we will use the formula,

$\theta=\tan^{-1}(b/a) = \theta=\tan^{-1}(5/7) = 35.54°$

Since 7−5i is in the fourth quadrant, so

θ=360°−35.54° = 324.46°

Hence, the polar form of 7-5i is represented by:

7−5i=8.6(cos 324.5°+i sin 324.5°)

EXAMPLE: Convert the given complex number in polar form : 1−i

Solution : Given, z=1−i

Let rcosθ=1 and rsinθ=−1

On squaring and adding, we obtain

$r^2\cos^2\theta+r^2\sin^2\theta=1^2+(-1)^2$

$\Rightarrow r^2(\cos^2\theta+\sin^2\theta)=2$

$\Rightarrow r^2=2$

$\Rightarrow r=2$ (since,r>0)

∴2cosθ=1 and 2sinθ=−1

∴θ=−4π (As θ lies in fourth quadrant.)

So, the polar form is

∴1−i=rcosθ+irsinθ=2cos(4−π)+i2sin(4−π)

=2[cos(4−π)+isin(4−π)]

De Moivre's Theorem

Mathematical Statement: For any real number x, we have

$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx)$

OR

$$(e^{i\theta})^n = e^{in\theta}$$

Where n is a positive integer and " i " is the imaginary part, and i = √(-1).

Also, assume $i^2 = -1$.

Uses of De Moivre's Theorem:-

To find the roots of complex numbers

If z is a complex number, and z = r(cos x + i sin x) [In polar form]

Then, the nth roots of z are:

$$r^{\frac{1}{n}}(cos(\frac{x+2k\pi}{n}) + i\, sin(\frac{x+2k\pi}{n}))$$

Where k = 0, 1, 2,….., (n − 1)

If k = 0, the above formula reduces to

$$r^{\frac{1}{n}}(cos(\frac{x}{n}) + i\, sin(\frac{x}{n}))$$

Q. Evaluate (1 + i)^{1000}.

Solution:

Let z = 1 + i

We have to represent z in the form of r(cos θ + i sin θ).

Here,

Argument = θ = arc(tan (1/1) = arc tan(1) = π/4

Absolute value = r
=(1)2+(1)2=2

Applying DeMoivre's theorem, we get

$z^{1000} = [\sqrt{2}\{\cos(\pi/4) + i \sin(\pi/4)\}]^{1000}$

$= 2^{1000} \{\cos(1000\pi/4) + i \sin(1000\pi/4)\}$

$= 2^{1000} \{1 + i (0)\}$

$= 2^{1000}$

Problem 1: Evaluate $(2 + 2i)^6$

Solution: Let z = 2 + 2i

Here, r = 2√2 and θ = 45 degrees

Since z lies in the first quadrant, sinθ and cosθ functions are positive.

Applying De Moivre's Theorem,

$z^6 = (2 + 2i)^6 = (2\sqrt{2})^6 [\cos 45^0 + i \sin 45^0]^6$

$= (2\sqrt{2})^6 [\cos 270^0 + i \sin 270^0]^6$

= – 512i

Problem 2: Express five fifth-roots of (√3 + i) in trigonometric form.

Solution:

We know, z = a + ib = r(cos x + i sin x)

Where

$$r = \sqrt{a^2 + b^2}$$

and tan x = (b/a)

So,

Here, r = 2 and θ = 30 degrees

Therefore, $z = 2[\cos(30^0 + 360^0 k) + i \sin \cos(30^0 + 360^0 k)]$

Applying the nth root theorem,

$z^{1/5} = \{2[\cos(30^0 + 360^0 k) + i \sin \cos(30^0 + 360^0 k)]\}^{1/5}$

$= 2^{1/5} [\cos((30^0 + 360^0 k)/5) + i \sin \cos((30^0 + 360^0 k)/5)]$ …(1)

Where k = 0,1,2,3,4

At k = 0; (1)=> $z_1 = 2^{1/5} [\cos 6^0 + i \sin 6^0]$

At k = 1; (1)=> $z_1 = 2^{1/5} [\cos 78^0 + i \sin 78^0]$

At k = 2; (1)=> $z_1 = 2^{1/5} [\cos 150^0 + i \sin 150^0]$

At k = 3; (1)=> $z_1 = 2^{1/5} [\cos 222^0 + i \sin 222^0]$

At k = 4; (1)=> $z_1 = 2^{1/5} [\cos 294^0 + i \sin 294^0]$

Problem 3: Express

$\left(\frac{\cos\theta + i \sin\theta}{\sin\theta + i \cos\theta}\right)^4$ **in a+ib form.**

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Solution:

$$\left(\frac{\cos\theta + i\,\sin\theta}{\sin\theta + i\,\cos\theta}\right)^4$$

$$= \frac{(\cos\,\theta + i\,\sin\theta)^4}{i^4(\,\cos\theta - i\sin\theta)^4}$$

= (cos 4θ + i sin 4θ) / (cos 4θ – i sin 4θ)

By rationalising the fraction, we have

= $(\cos 4\theta + i \sin 4\theta)^2 / (\cos^2 4\theta - i \sin^2 4\theta)$

= cos 8θ + i sin 8θ

Example :-

$$\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18}.$$

Solution

We have, $\sin\frac{\pi}{6} + i\cos\frac{\pi}{6} = i\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$.

Raising to the power 18 on both sides gives,

$$\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18} = (i)^{18}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{18}$$

$$= (-1)\left(\cos\frac{18\pi}{6} - i\sin\frac{18\pi}{6}\right)$$

$$= -(\cos 3\pi - i\sin 3\pi) = 1 + 0i.$$

Therefore, $\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18} = 1.$

Q.

Simplify $\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^{30}.$

Solution:-

Let $z = \cos 2\theta + i\sin 2\theta$.

As $|z| = |z|^2 = z\bar{z} = 1$, we get $\bar{z} = \frac{1}{z} = \cos 2\theta - i\sin 2\theta$.

Therefore, $\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta} = \frac{1+z}{1+\frac{1}{z}} = \frac{(1+z)z}{z+1} = z$.

Therefore, $\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^{30} = z^{30} = (\cos 2\theta + i\sin 2\theta)^{30}$

$= \cos 60\theta + i\sin 60\theta$.