Unit 1: ALGEBRA-1

1. Sequence:

- **Definition:** A sequence is an ordered list of numbers, objects, or terms, typically arranged in a specific pattern or according to a rule. Each element in the sequence is called a term.
- **Example:** The sequence of natural numbers {1, 2, 3, 4, 5, ...} is a commonly known example. Another example is the Fibonacci sequence {1, 1, 2, 3, 5, 8, ...}, where each term is the sum of the two preceding terms.

2. Series:

- **Definition:** A series is the sum of the terms in a sequence. It's formed by adding up all the terms in a sequence.
- **Example:** If we have the sequence {1, 2, 3, 4, 5, ...}, then the series would be 1 + 2 + 3 + 4 + 5 + ... = ∞ (infinity) because the sequence of natural numbers extends indefinitely.

3. Progression:

- **Definition:** The term "progression" is a general term used to describe a sequence or series of numbers or objects that follow a particular order, rule, or pattern. It includes various types of progressions, such as arithmetic, geometric, and harmonic progressions.
- **Example:** An arithmetic progression (AP) and a geometric progression (GP) are specific types of progressions. These types have their own distinct rules.

4. Arithmetic Progression (AP):

- **Definition:** An arithmetic progression is a sequence of numbers in which each term is obtained by adding a fixed constant (called the common difference) to the previous term.
- **Example:** Consider the sequence {2, 5, 8, 11, 14, ...}. Here, the common difference is 3 because you add 3 to each term to get the next one. This is an arithmetic progression.

In the example of the arithmetic progression {2, 5, 8, 11, 14, ...}, you can see that the common difference between each consecutive term is 3. So, the next term is obtained by adding 3 to the previous term:

2 + 3 = 5

5 + 3 = 8

8 + 3 = 11

11 + 3 = 14

n^{th} Term of an AP

Assume that a_1, a_2, a_3,… be an arithmetic progression (AP), in which first term a_1 is equal to "a" and the common difference is taken as "d", then the second term, third term, etc can be calculated as follows:

Second term, a_2 = a+d

Third term, a_3 = (a+d)+d = a+2d,

Fourth term, a_4 = (a+2d)+d = a+3d, and so on.

Therefore, the nth term of an AP (a_n) with the first term "a" and common difference "d" is given by the formula:

n^{th} term of an AP, $a_n = a+(n-1)d$.

(**Note**: The nth term of an AP (a_n) is sometimes called the general term of an AP, and also the last term in a sequence is sometimes denoted by "l".)

Also, read: Sum of N terms of AP

n^{th} Term of an AP

Go through the following examples to understand the procedure for finding the nth term of an AP.

Example 1:
Determine the 10th term of an AP 2, 7, 12, ….
Solution:
Given arithmetic progression (AP) is 2, 7, 12, …
Here, the first term, a = 2.
Common difference, d = 7-2 = 5
n=10.
The formula to find the nth term of an AP, $a_n = a+(n-1)d$
Now, substitute the values in the formula, we get
$a_{10} = 2 + (10-1)5$

$a_{10} = 2 + (9)5$

$a_{10} = 2+45$

$a_{10} = 47.$

Hence, the 10th term of an AP 2, 7, 12, … is 47.

Example 2:

The third term of an AP is 5 and the 7th term of an AP is 9. Find the arithmetic progression (AP).

Solution:

Given that, Third term of AP = 5

Seventh term of AP = 9

(i.e) $a_3 = a+(3-1)d = a+2d = 5$ …(1)

$a_7 = a+(7-1)d = a+6d = 9$ …(2)

Now, solve the equations (1) and (2), we get

a=3 and d = 1.

Therefore, the first term is 3 and the common difference is 1.

Therefore, the arithmetic progression (AP) is 3, 4, 5, 6, 7, 8, 9, ….

Example 3:

How many two-digit numbers are divisible by 3?

Solution:

The sequence of two-digit numbers which are divisible by 3 are:

12, 15, 18, 21, …, 99.

To find whether the given sequence is an AP, find the common difference.

The common difference (d) of the above-given sequence is 3, and hence, the given sequence is an Arithmetic progression (AP).

Hence, a = 12, d = 3, $a_n = 99$.

Now, we have to find the value of "n".

Now, substitute the values in the formula, $a_n = a+(n-1)d$, we get

$99 = 12+(n-1)3$

$99 = 12+3n-3$

$99-12+3 = 3n$

$3n = 90$

$n= 90/3$

$n=30.$

Hence, there are 30 two-digit numbers that are divisible by 3.

Sum of n Terms of an A.P :-

The sum of n terms of AP is the sum(addition) of first n terms of the arithmetic sequence. It is equal to n divided by 2 times the sum of twice the first term – 'a' and the product of the difference between second and first term-'d' also known as common difference, and (n-1), where n is numbers of terms to be added.

Sum of n terms of AP = n/2[2a + (n – 1)d]=n/2(a+l)

(where l is the last term of an A.P)

For example:

- 1, 4, 9, 16, 25, 36, 49 ……….625 represents a sequence of squares of natural numbers till 25.
- 3, 7, 11, 15, 19,………..87 forms another sequence, where each of the terms exceeds the preceding term by 4.

If all the terms of a progression except the first one exceeds the preceding term by a fixed number, then the progression is called arithmetic progression. If a is the first term of a finite AP and d is a common difference, then AP is written as – ***a, a+d, a+2d, ………, a+(n-1)d.***

Note: Before learning how to derive a formula to get the sum of n terms in an AP, try this activity:

- Try to get the sum of the first 100 natural numbers without using any formula.:

- The sum of the number can be represented as

Sum = 1+2+3+4+……………+ 97 + 98 + 99 + 100——————————— (1)

- Even if the order of the numbers is reversed, their sum remains the same.

Sum = 100 + 99 + 98 + 97 + ………..+ 4 + 3 + 2 + 1——————————— (2)

Adding equations 1 and 2, we get

- 2 × Sum = (100+ 1) + (99+2) + (98+3)+ (97 +4)+ ………..(4+97)+(3+98)+(2+99)+(1+100)
-
- 2 × Sum = 101 + 101 + 101 + 101 + ………..(4+97)+(3+98)+(2+99)+(1+100)
- 2 ×Sum = 101 (1 + 1 + 1 + …..100 terms)
- 2 × Sum = 101 (100)
- Sum = {101 × 100}/{2}
- Sum = 5050

Arithmetic Mean(A.M) between Two Numbers

Consider any two numbers, say m and n, and P be the arithmetic mean between two numbers.

The sequence will be m, P, and n in AP.

P – m = n – P

P = (n + m)/2 = (Sum of the numbers)/(number of terms).

To insert n A.M between two numbers :-

Let a, A_1, A_2, A_3,……A_n, b is in A.P. Here b is the (n + 2)th term

So, b = a + [(n + 2) – 1]d

b = a + (n + 1)d

b – a = (n + 1)d $d = \frac{b-a}{n+1}$

Thus 'n' arithmetic means between ‘a’ and ‘b’ are as follows $A_1 = a + d = a + \frac{b-a}{n+1}$

$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$ $A_3 = a + 3d = a + \frac{3(b-a)}{n+1}$ $A_n = a + nd = a + \frac{n(b-a)}{n+1}$

Example 1 :-Insert three arithmetic means between 8 and 26.

Solution :-

Let three arithmetic number inserted will be A1, A2 and A3 between 8and 26.

8, A_1, A_2, A_3, 26 are in A.P. Then

A = 8 and b = 26 and n = 5

$a_n = a + (n - 1)d$

26 = 8 + 4d

18 = 4d

$\frac{18}{4} = \frac{4d}{4}$

∴ d = 4.5

A1 = a + d = 8 + 4.5 = 12.5

A2 = a + 2d = 8 + 2 × 4.5 = 17

A3 = a + 3d = 8 + 3 × 13.5 = 21.5

Thus the three arithmetic means between 8 and 26 are 12.5, 17 and 21.5.

Example 2 :-Insert 6 number 3 and 24 such that the resulting sequence is and A.P.

Solution :-

Let A1, A2, A3, A4, A5 and A6 be six number between 3 and 24 such that 3, A1, A2, A3, A4, A5, A6, 24 are in A.P. Here, a = 3, b = 24, n = 8.
Therefore, 24 = 3 + (8 – 1)d, so that d = 3.
Thus,A1 = a + d = 3 + 3 = 6;
A2 = a + 2d = 3 + 2 × 3 = 9;
A3 = a + 3d = 3 + 3 × 3 = 12;
A4 = a + 4d = 3 + 4 × 3 = 15;
A5 = a + 5d = 3 + 5 × 3 = 18;
A6 = a + 6d = 3 + 6 × 3 = 21.
Hence, six numbers between 3 and 24 are 6, 9, 12, 15, 18 and 21.

Sum of n arithmetic mean between two numbers:-

Let a, A_1, A_2, A_3,......A_n, b is in A.P.

Sum of n A.M between a and b =n (a + b)/2

Q. The sequence 28, 22, x, y, 4 is an AP. Find the values of x and y.

Solution:

Given AP: 28, 22, x, y, 4

Here, first term, a = 28

Common difference, d = 22 – 28 = -6

Hence, x = 22 – 6 = 16

y = 16 – 6 = 10.

Hence, the values of x and y are 16 and 10, respectively.

Q. Which term of AP 27, 24, 21, … is 0?

Solution:

Given AP: 27, 24, 21, …

Here, a = 27

d = 24 – 27 = -3.

Also given that a_n = 0

Now, we have to find the value of n.

Hence, a_n = a+(n-1)d

0 = 27 +(n-1)(-3)

0 = 27 -3n +3

0 = 30 – 3n

3n = 30

Hence, n = 10

Therefore, the 10th term of AP is 0.

Q. The general term of a sequence is given by a_n = -4n + 15. Is the sequence an A. P.? If so, find its 15^{th} term and the common difference.

$a_n = -4n+15$

$a_k = -4k+15$

$a_{k+1} = -4(k+1)+15$

Now

$a_{k+1} - a_k = -4(k+1)+15-[-4k+15] = -4$

Since difference between two terms constant. It is a AP

$a_{15} = -4(15)+15 = -45$.

Q. If the first, second and last terms of the AP are 5, 9, 101, respectively, find the total number of terms in the AP.

Solution:

Given: First term, a = 5

Common difference, d = 9 – 5 = 4

Last term, $a_n = 101$

Now, we have to find the value of "n".

Hence, $a_n = a+(n-1)d$

Substituting the values, we get

5+(n-1)4 = 101

5+4n-4 = 101

4n+1= 101

4n = 100

n=100/4 = 25

Hence, the number of terms in the AP is 25.

Q. If the first term of an AP is 67 and the common difference is -13, find the sum of the first 20 terms.

Solution: Here, a = 67 and d= -13

$S_n = n/2[2a+(n-1)d]$

$S_{20} = 20/2[2\times67+(20-1)(-13)]$

$S_{20} = 10[134 - 247]$

$S_{20} = -1130$

So, the sum of the first 20 terms is -1130.

***Q.* The sum of n and n-1 terms of an AP is 441 and 356, respectively. If the first term of the AP is 13 and the common difference is equal to the number of terms, find the common difference of the AP.**

Solution: The sum of n terms $S_n = 441$

Similarly, $S_{n-1} = 356$

a = 13

d= n

For an AP, $S_n = (n/2)[2a+(n-1)d]$

Putting n = n-1 in above equation,

l is the last term. It is also denoted by a_n. The result obtained is:

$S_n - S_{n-1} = a_n$

So, 441-356 = a_n

a_n = 85 = 13+(n-1)d

Since d=n,

n(n-1) = 72

$\Rightarrow n^2 - n - 72 = 0$

Solving by factorization method,

$n^2 - 9n + 8n - 72 = 0$

(n-9)(n+8)=0

So, n= 9 or -8

Since number of terms can't be negative,

n= d = 9

Find the A.M. between:

(i) 7 and 13

(ii) 12 and -8

(iii) (x−y) **and** (x+y)

Solution

(i) 7 and 13

Let A be the arithmetic mean of 7 and 13

Then,

7, A, 13 are in A.P.

$\Rightarrow A-7=13-A \Rightarrow A=13+72=10$

∴ **A.M. is 10.**

(ii) 12 and -8

Let A be the arithmetic mean of 12 and -8

Then,

12, A, -8 are in A.P.

$\Rightarrow A-12=-8-A \Rightarrow A=12+(-8)2=2$

∴ **A.M. is 2.**

(iii) $(x-y)$ **and** $(x+y)$

Let A be the arithmetic mean of (x - y) and (x + y)

Then,

$(x-y), A, (x+y)$ **are in A.P**

⇒A−(x−y)=(x+y)−A⇒A=(x−y)+(x+y)2=2x2=2

∴ **A.M is x.**

Q. Insert 4 A.M.s between 4 and 19.

Solution

Let A1,A2,A3,A4 **be the 4 A.M.s between 4 and 19.**

Then,

4,A1,A2,A3,A4,19 **are in A.P. of 6 terms.**

An=a+(n−1)da6=19=4+(6−1)d

Or d = 3 ...... (i)

Now,

A1=a+d=4+3=7A2=A1+d=7+3=10A3=A2+d=10+3=13A4=A3+d=13+3=16

The 4 A.M.s between 4 and 19 are 7, 10, 13, 16.

Q. Find the of Sum of 8 A.M between 5 and 11 ?

Sol. Sum of 8 A.M between 5 and 11 is = 8 x (5+11)/2

= 8 x 8

= 64

Q. The sum of three consecutive terms of an A. P. is 21 and the sum of their squares is 165. Find these terms.

Solution

Let the terms are a - d, a, a + d.

Sum = a - d + a + a + d = 3 a

3 a = 21

a = 7

Second condition,

a2 **+** d2 **- 2ad +** a2 **+** a2 **+** d2 **+ 2ad = 165**

3a2 **+ 2**d2 **= 165**

$49\times3 + 2d^2 = 165$

$147 + 2d^2 = 165$

$2d^2 = 18$

$d^2 = 9$

$d = 3$

So the series is 4,7,10

Q. Find four consecutive terms in A.P whose sum is 20 and the sum of whose squares is 120.

Solution

Let the A.P be a-3d, a-d, a+d, a+3d

so according to the question

$a-3d+a-d + a+d + a+3d = 20$

$4a = 20$

$a=5$

also

$(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120$

$(5-3d)^2+(5-d)^2+(5+d)^2+(5+3d)=120$

$25+9d^2-30d+25+d^2-10d+25+d^2+10d+25+9d^2+30d=120$

$100+20d^2=120$

$20d^2=120-100$

$d^2=1$

$d=1$

therefore the AP will be 5-3(1), 5-1, 5+1, 5+3(1)

2,4,6,8.

Geometric Progression (G.P.): A geometric progression, often abbreviated as G.P., is a sequence of numbers in which each term after the first is obtained by multiplying the previous term by a fixed, non-zero number called the common ratio (r). In a G.P., the ratio of any two consecutive terms is constant.

General Term of a G.P.: The general term (a_n) of a geometric progression can be expressed as:

$a_n = a \cdot r(n-1)$

Where:

- a_n is the nth term you want to find.
- $a1 = a$ is the first term of the G.P.
- r is the common ratio.
- n is the position of the term you want to find.

Sum of the First n Terms of a G.P. (Finite Sum): The sum of the first n terms of a geometric progression can be calculated using the formula:

$S_n = a(1 - r^n) / (1 - r)$, when $r \neq 1$.

Where:

- Sn is the sum of the first n terms.
- a is the first term of the G.P.
- r is the common ratio.
- n is the number of terms for which you want to find the sum.

Sum of an Infinite Geometric Progression: If a geometric progression extends infinitely, and its common ratio (r) is between -1 and 1 (inclusive), you can find the sum of the infinite terms using the following formula:

$S_n = a / (1 - r)$, when $|r| < 1$

Q. If the first term is 10 and the common ratio of a GP is 3, then write the first five terms of GP.

Solution: Given,

First term, a = 10

Common ratio, r = 3

We know the general form of GP for first five terms is given by:

a, ar, ar^2, ar^3, ar^4

$a = 10$

$ar = 10 \times 3 = 30$

$ar^2 = 10 \times 3^2 = 10 \times 9 = 90$

$ar^3 = 10 \times 3^3 = 270$

$ar^4 = 10 \times 3^4 = 810$

Therefore, the first five terms of GP with 10 as the first term and 3 as the common ratio are:

10, 30, 90, 270 and 810

Q. Find the sum of GP: 10, 30, 90, 270 and 810, using formula.

Solution: Given GP is 10, 30, 90, 270 and 810

First term, a = 10

Common ratio, r = 30/10 = 3 > 1

Number of terms, n = 5

Sum of GP is given by;

$S_n = a[(r^n – 1)/(r – 1)]$

$S_5 = 10[(3^5 – 1)/(3 – 1)]$

= 10[(243 – 1)/2]

= 10[242/2]

= 10 × 121

= 1210

Check: 10 + 30 + 90 + 270 + 810 = 1210

Q. If 2, 4, 8,…., is the GP, then find its 10th term.

Solution: The nth term of GP is given by:

2, 4, 8,….

Here, a = 2 and r = 4/2 = 2

$a_n = ar^{n-1}$

Therefore,

$a_{10} = 2 \times 2^{10 – 1}$

$= 2 \times 2^9$

= 1024

Q. If the sum of the geometric series 1+4+16+64+… **is** 5461**. Then the number of terms is**

Solution

Let the required number of terms be n**.**

Here, a=1, r=4>1 **and** Sn=5461**.**

∴Sn=a(rn−1)(r−1)

⇒1×(4n−1)(4−1)=5461

⇒(4n−1)=16383

⇒4n=16384=47

⇒n=7

Hence, the required number of terms is 7.

Q.Sum the series 5 + 55 + 555 + ... to n terms.

Solution

We have 5 + 55 + 555 + ... to n terms

= 5 × {1 + 11 + 111 + ... to n terms}

=59 × **{9 + 99 + 999 + ... to n terms}**

=59×{(10−1)+(102−1)+(103−1)+⋯to n terms}

=59×{(10+102+103+⋯to n terms)−n}

=59×{10×(10n−1)(10−1)−n}=581×(10n+1−9n−10)

Hence, the required sum is 581×(10n+1−9n−10)

Q. Find the sum of the sequence 1, (12),(14), ... till infinity ?

Solution

Sum of infinite terms of a G.P. whose first term is a and common ratio is r is a(1−r)

Then, the required sum will be 1/(1−(1/2))=2.

Q. Find the 12th term of a G.P. whose 8th term is 192, and the common ratio is 2.

Solution:

Given,

The common ratio of the G.P., $r = 2$

And, let a be the first term of the G.P.

Now,

$a_8 = ar^{8-1} = ar^7$

$ar^7 = 192$

$a(2)^7 = 192$

$a(2)^7 = (2)^6\,(3)$

So,

$$a = \frac{(2)^6 \times 3}{(2)^7} = \frac{3}{2}$$

Hence,

$$a_{12} = ar^{12-1} = \left(\frac{3}{2}\right)(2)^{11} = (3)(2)^{10} = 3072$$

Q. The 4th term of a G.P. is the square of its second term, and the first term is –3. Determine its 7th term.

Solution:

Let's consider *a* to be the first term and *r* to be the common ratio of the G.P.

Given, $a = -3$

And we know that,

$a_n = ar^{n-1}$

So, $a_4 = ar^3 = (-3)\ r^3$

$a_2 = a\ r^1 = (-3)\ r$

Then, from the question, we have

$(-3)\ r^3 = [(-3)\ r]^2$

$\Rightarrow -3r^3 = 9\ r^2$

$\Rightarrow r = -3$

$a_7 = a\ r^{7-1} = a\ r^6 = (-3)\ (-3)^6 = -\ (3)^7 = -2187$

Therefore, the seventh term of the G.P. is –2187.

Q. For what values of *x*, the numbers -2/7, x, -7/2 are in G.P?

Solution:

The given numbers are -2/7, x, -7/2

Common ratio = x/(-2/7) = -7x/2

Also, common ratio = (-7/2)/x = -7/2x

$$\therefore \frac{-7x}{2} = \frac{-7}{2x}$$

$$x^2 = \frac{-2\times 7}{-2\times 7} = 1$$

$$x = \sqrt{1}$$

$$x = \pm 1$$

Therefore, for *x* = ± 1, the given numbers will be in G.P.

GEOMETRIC MEAN:-

If three terms are in g.p., then the middle term is called the geometric mean (g.m.) between the two. So if a, b, c are in g.p., then **b = √ac** is the geometric mean of a and c.

To insert n geometric means between two numbers a and b:-

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- If a_1, a_2, ……, a_n are non-zero positive numbers, then their G.M.(G) is given by G = $(a_1a_2a_3, ……, a_n)^{1/n}$. If G_1, G_2, …… G_n are n geometric means between and a and b then a, G_1, G_2, ……, G_n b will be a G.P. Here $b = ar^{n+1}$.

  ⇒ $r = {}^{n+1}\sqrt{b/a} \Rightarrow G_1 = a\,{}^{n+1}\sqrt{b/a}$, $G_2 = a({}^{n+1}\sqrt{b/a})^2$,…, $G_n = a({}^{n+1}\sqrt{b/a})^n$.

The product of n geometric means between two numbers a and b:-

Let x1,x2,x3,…………xn be n G.M. between a and b then
a,x1,x2,x3,…….xn, b
$b=ax+2=ar^{n+1}$
Product of n GM
=x1,x2….xn
$=(ab)^{n/2}$

Q. Find the geometric mean of 4 and 3.

Solution: Using the formula for G.M., the geometric mean of 4 and 3 will be:

Geometric Mean will be $\sqrt{(4\times3)}$

$= 2\sqrt{3}$

So, GM = 3.46

Q. What is the geometric mean of 4, 8, 3, 9 and 17?

Solution:

Step 1: n = 5 is the total number of values. Now, find 1/n.

1/5 = 0.2.

Step 2: Find geometric mean using the formula:

$(4 \times 8 \times 3 \times 9 \times 17)^{0.2}$

So, geometric mean = 6.814

Q. Four geometric means are inserted between 5 and 160. Find the 2nd geometric mean.

Solution

$5\times r^5=160 \rightarrow r^5=32 \Rightarrow r=2$

Thus, the series would be 5, 10, 20, 40, 80, 160. The second geometric mean between 5 and 160 in this case would be 20.

Q. If 34 and 16 are the arithmetic mean and geometric mean of two positive numbers respectively, Find the numbers?

Solution

Let the two numbers be m and n. Then

Arithmetic Mean = 34

$\Rightarrow (m+n)/2=34$

$\Rightarrow m+n=68$

And

Geometric Mean = 16

$\sqrt{mn}=16$

$\Rightarrow mn=256$ .... **(i)**

Therefore, $(m-n)2=(m+n)2-4mn$

$\Rightarrow (m-n)2=(68)2-4\times 256=3600$

$\Rightarrow m-n=60$ .... **(ii)**

On solving (i) and (ii), we get m = 64 and n = 4.

Hence, the required numbers are 64 and 4.

Q. If the product of three numbers in GP is 216 and the sum of their products in pairs is 156, find the numbers.

Solution

Let the required numbers be a/r**, a and ar. Then,**

$a/r\times a\times ar=216\Rightarrow a3=216=6^3\Rightarrow a=6$

And, $a/r\times a+a\times ar+a/r\times ar=156$

$\Rightarrow \quad a^2(1/r+r+1)=156\Rightarrow (6^2)(1+r+r^2)=156r \qquad [\because \quad a=6]$

$\Rightarrow \quad 36(r^2+r+1)=156r\Rightarrow 3(r^2+r+1)=13r$

$\Rightarrow \quad 3r^2-10r+3=0\Rightarrow (3r-1)(r-3)=0\Rightarrow r=13$ **or** $r=3$

So, the required numbers are 18, 6, 2 or 2, 6, 18.

Q. If products of three terms of a GP is 216 and sum of their products taken in pairs is 156, then find the numbers.

Solution

Let the no. be

α/r,α,αr

α/r×α×αr=216

$\alpha^3=(6)^3 \Rightarrow \alpha=6$

α/2=6/3=2

αr=18(r=3)

Hence no. are 2,6 **and**18**.**

$\alpha^2/r\times\alpha^2\times\alpha^2 r=156$

36[r+1/r+1]=156

r+1/r=133−1

r+1/r=103

$(r^2+1)/r=103$

$3r^3-10r+3=0$

$3r^3-9r-r+3=0$

3r(r−3)−1(r−3)=0

r=13,r=3

Binomial Theorem:-

The binomial theorem is the method of expanding an expression that has been raised to any finite power. A binomial theorem is a powerful tool of expansion which has applications in Algebra, probability, etc.

Binomial Expression: A binomial expression is an algebraic expression that contains two dissimilar terms. Eg.., a + b, $a^3 + b^3$, etc.

Binomial Theorem for positive, negative & fraction index:-

We have $(x + y)^n = nC_0\, x^n + nC_1\, x^{n-1} . y + nC_2\, x^{n-2} . y^2 + \ldots + nC_n\, y^n$

General Term = $T_{r+1} = nC_r\, x^{n-r} . y^r$

$(1 + x)^n = 1 + n x + [n(n - 1)/2!]\, x^2 + [n(n - 1)(n - 2)/3!]\, x^3 + \ldots$

Where (n_{Cr}=n! / r! n-r!)

Some other useful expansions:

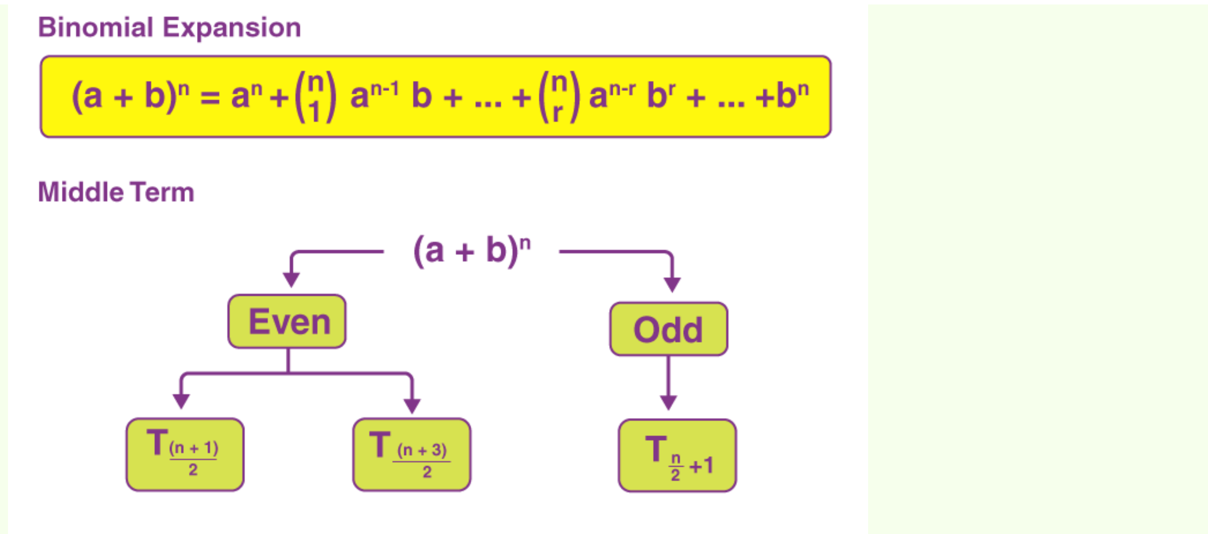
- $(x + y)^n + (x-y)^n = 2[C_0\, x^n + C_2\, x^{n-1}\, y^2 + C_4\, x^{n-4}\, y^4 + \ldots]$
- $(x + y)^n - (x-y)^n = 2[C_1\, x^{n-1}\, y + C_3\, x^{n-3}\, y^3 + C_5\, x^{n-5}\, y^5 + \ldots]$

- $(1 + x)^n = {}^n\Sigma_{r-0}\ nC_r \,.\, x^r = [C_0 + C_1\, x + C_2\, x^2 + \dots\, C_n\, x_n]$
- $(1+x)^n + (1 - x)^n = 2[C_0 + C_2\, x^2 + C_4\, x^4 + \dots]$
- $(1+x)^n - (1-x)^n = 2[C_1\, x + C_3\, x^3 + C_5\, x^5 + \dots]$
- The number of terms in the expansion of $(x + a)^n + (x-a)^n$ is $(n+2)/2$ if "n" is even or $(n+1)/2$ if "n" is odd.
- The number of terms in the expansion of $(x + a)^n - (x-a)^n$ is $(n/2)$ if "n" is even or $(n+1)/2$ if "n" is odd.

Properties of Binomial Coefficients

Binomial coefficients refer to the integers, which are coefficients in the binomial theorem. Some of the most important properties of binomial coefficients are:

- $C_0 + C_1 + C_2 + \dots + C_n = 2^n$
- $C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$
- $C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n \,.\, nC_n = 0$
- $nC_1 + 2.nC_2 + 3.nC_3 + \dots + n.nC_n = n.2^{n-1}$
- $C_1 - 2C_2 + 3C_3 - 4C_4 + \dots + (-1)^{n-1}\, C_n = 0$ for $n > 1$
- $C_0^2 + C_1^2 + C_2^2 + \dots C_n^2 = [(2n)!/\,(n!)^2]$



Q : $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5$

Sol:

We have

$(x + y)^5 + (x - y)^5 = 2[5C_0\ \ x^5 + 5C_2\ \ x^3\, y^2\ \ +\ \ 5C_4\ xy^4]$

$= 2(x^5 + 10\, x^3\, y^2 + 5xy^4)$

Now $(\sqrt{2} + 1)^5 + (\sqrt{2} - 1)^5 = 2[(\sqrt{2})^5 + 10(\sqrt{2})^3(1)^2 + 5(\sqrt{2})(1)^4]$

$=58\sqrt{2}$

Q: Expand the expression $(2x\text{-}3)^6$ using the binomial theorem.

Solution: Given Expression: $(2x\text{-}3)^6$

By using the binomial theorem, the expression $(2x-3)^6$ can be expanded as follows:

$(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6$

$(2x-3)^6 = 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) + 15(4x^2)(81) - 6(2x)(243) + 729$

$(2x-3)^6 = 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Thus, the binomial expansion for the given expression $(2x-3)^6$ is $64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$.

Q: Evaluate $(101)^4$ using the binomial theorem

Solution:

Given: $(101)^{4.}$

Here, 101 can be written as the sum or the difference of two numbers, such that the binomial theorem can be applied.

Therefore, 101 = 100+1

Hence, $(101)^4 = (100+1)^4$

Now, by applying the binomial theorem, we get:

$(101)^4 = (100+1)^4 = {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4$

$(101)^4 = (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4$

$(101)^4 = 100000000 + 4000000 + 60000 + 400 + 1$

$(101)^4 = 104060401$

Hence, the value of $(101)^4$ is 104060401.

Q:

Using the binomial theorem, show that $6^n - 5n$ always leaves remainder 1 when divided by 25

Solution: Assume that, for any two numbers, say x and y, we can find numbers q and r such that x = yq + r, then we say that b divides x with q as quotient and r as remainder. Thus, in order to show that $6^n - 5n$ leaves remainder 1 when divided by 25, we should prove that $6^n - 5n = 25k + 1$, where k is some natural number.

We know that,

$(1 + a)^n = {}^nC_0 + {}^nC_1\, a + {}^nC_2\, a^2 + \ldots + {}^nC_n\, a^n$

Now for a=5, we get:

$(1 + 5)^n = {}^nC_0 + {}^nC_1\, 5 + {}^nC_2\, (5)^2 + \ldots + {}^nC_n\, 5^n$

Now the above form can be weitten as:

$6^n = 1 + 5n + 5^2\, {}^nC_2 + 5^3\, {}^nC_3 + \ldots + 5^n$

Now, bring 5n to the L.H.S, we get

$6^n - 5n = 1 + 5^2\ {}^nC_2 + 5^3\ {}^nC_3 + \ldots. + 5^n$

$6^n - 5n = 1 + 5^2\ ({}^nC_2 + 5\ {}^nC_3 + \ldots. + 5^{n-2})$

$6^n - 5n = 1 + 25\ ({}^nC_2 + 5\ {}^nC_3 + \ldots. + 5^{n-2})$

$6^n - 5n = 1 + 25\ k$ (where $k = {}^nC_2 + 5\ {}^nC_3 + \ldots. + 5^{n-2}$)

The above form proves that, when $6^n - 5n$ is divided by 25, it leaves the remainder 1.

Hence, the given statement is proved.

Q:

Find the value of r, If the coefficients of $(r - 5)^{th}$ and $(2r - 1)^{th}$ terms in the expansion of $(1 + x)^{34}$ are equal.

Solution:

For the given condition, the coefficients of $(r - 5)^{th}$ and $(2r - 1)^{th}$ terms of the expansion $(1 + x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$ respectively.

Since the given terms in the expansion are equal,

${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

From this, we can write it as either

r-6=2r-2

(or)

r-6=34 -(2r-2) [We know that, if ${}^nC_r = {}^nC_p$, then either r = p or r = n – p]

So, we get either r = – 4 or r = 14.

We know that r being a natural number, the value of r = – 4 is not possible.

Hence, the value of r is14.

Q. Find y if the 17th and 18th terms of the expansion $(2 + y)^{50}$ are equal.

Solution:

(r + 1)th term of the expansion of $(a + b)^n = T_{r+1} = {}^nC_r\ a^{n-r}\ b^r$

Here, a = 2, b = y, n = 50

17th term = (16 + 1)th term, i.e. r = 16

$T_{17} = T_{16+1} = {}^{50}C_{16}\ (2)^{50-16}\ y^{16}$

$= {}^{50}C_{16}\ (2)^{34}\ (y)^{16}$

Similarly, 18th term = (17 + 1)th term, i.e. r = 17

$T_{18} = T_{17+1} = {}^{50}C_{17}\ (2)^{50-17}\ y^{17}$

$= {}^{50}C_{13}\ (2)^{33}\ (y)^{17}$

According to the given,

$T_{17} = T_{18}$

${}^{50}C_{16}\ (2)^{34}\ (y)^{16} = {}^{50}C_{13}\ (2)^{33}\ (y)^{17}$

$\Rightarrow y = (2 \times {}^{50}C_{16}) / {}^{50}C_{17}$

= 2 × [50!/(16! 34!)] × [(17! 33!)/50!]

= 2 × [(17 × 16! × 33!)/ (16! × 34 × 33!)]

= 1

Therefore, y = 1

Q. Find the middle term(s) in the expansion of $(x + 2y)^9$.

Solution:

Given: $(x + 2y)^9$

Comparing with $(a + b)^n$, we get;

a + x, b = 2y and n = 9 (odd)

As the value of n is odd, there will be two middle terms.

(n + 1)/2 = (9 + 1)/2 = 10/2 = 5

(n + 3)/2 = (9 + 3)/2 = 12/2 = 6

Thus, 5th and 6th terms are the middle terms.

$T_5 = T_{4+1} = {}^9C_4 (x)^{9-4} (2y)^4$ {since $T_{r+1} = {}^nC_r a^{n-r} b^r$}

$= 126 x^5 (2)^4 (y)^4$

$= (126 × 16) x^5 y^4$

$= 2016 x^5 y^4$

Also, $T_6 = T_{5+1} = {}^9C_5 (x)^{9-5} (2y)^5$

$= 126 x^4 (2)^5 (y)^5$

$= (126 × 32) x^4 y^5$

$= 4032 x^4 y^5$

Therefore, $2016 x^5 y^4$ and $4032 x^4 y^5$ are the middle terms in the expansion of $(x + 2y)^9$.

Q. Find the middle term(s) in the expansion of $(x + 3)^8$.

Solution:

Given: $(x + 3)^8$

Comparing with $(a + b)^n$, we get;

a + x, b = 3 and n = 8 (even)

So, there will be only one middle term.

(n/2) + 1 = (8/2) + 1 = 4 + 1 = 5

Thus, 5th term is the middle term.

$T_5 = T_{4+1} = {}^8C_4 (x)^{8-4} (3)^4$ {since $T_{r+1} = {}^nC_r a_{n-r} b^r$}

$= 70 x^4 × 81$

= (70 × 81) x4

$= 5670 x^4$

Hence, $5670 x^4$ is the middle term of the expansion $(x + 3)^8$.

Determinants **:-**Determinants are mathematical values associated with square matrices. They provide important information about the properties of the matrix and its solutions to systems of linear equations. The determinant of a matrix is denoted by "det(A)" or "|A|" for a matrix A.

Determinant of a 2 x 2 Matrix

Suppose, A = [aij] is a 2 x 2 matrix (order two matrix), such that;

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then the determinant of A is defined as:

Det (A) =

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Det (A) = $a_{11}.a_{22} - a_{12}.a_{21}$

Or

$|A| = a_{11}.a_{22} - a_{12}.a_{21}$

This is the determinant formula for matrix of order two.

Q.

$$\det\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} = (2)(4)-(5)(3)$$
$$= 8 - 15$$
$$= -7$$

Find determinant of $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

$$|A| = 3 \times 4 - 1 \times 2$$
$$= 12 - 2$$
$$= 10$$

Determinant of a 3 x 3 matrix

Let's suppose you are given a square matrix A where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let's calculate the determinant of matrix A, i.e., |A|.

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Find determinant of $B = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -2 \end{bmatrix}$

$|B| = 9 \times \begin{vmatrix} -1 & 6 \\ 0 & -2 \end{vmatrix} - 2 \times \begin{vmatrix} 5 & 6 \\ 4 & -2 \end{vmatrix} + 1 \times \begin{vmatrix} 5 & -1 \\ 4 & 0 \end{vmatrix}$

$= 9\,((-1) \times (-2) - 0 \times 6) - 2\,(5 \times (-2) - 4 \times 6) + 1\,(5 \times 0 - 4 \times (-1))$

$= 9\,(2 - 0) - 2\,(-10 - 24) + 1\,(0 + 4)$

$= 9 \times 2 - 2 \times (-34) + 1 \times 4$

$= 18 + 68 + 4$

$= 90$

Expand along column 1:

$\begin{vmatrix} -8 & -7 & 6 \\ -2 & 1 & 1 \\ 0 & 7 & 7 \end{vmatrix} =$

$-8\begin{vmatrix} 1 & 1 \\ 7 & 7 \end{vmatrix} + 2\begin{vmatrix} -7 & 6 \\ 7 & 7 \end{vmatrix} + 0\begin{vmatrix} -7 & 6 \\ 1 & 1 \end{vmatrix} =$

$-8(0) + 2(-91) + 0(-13) =$

$0 - 182 + 0 =$

-182

Properties of Determinants :-

1. Size and Square Matrix: Determinants are defined for square matrices only, meaning matrices with an equal number of rows and columns.

2. Scalar Value: A determinant is a scalar value, not a matrix.

3. Denoted as "det": The determinant of a matrix A is typically denoted as "det(A)" or "|A|".

4. Switching Rows/Columns: Interchanging any two rows or columns of a matrix changes the sign of its determinant.

5. Scalar Multiplication: If you multiply all the elements in a row or column of a matrix by a scalar k, the determinant is multiplied by k.

6. Row Operations: Performing row operations (e.g., adding a multiple of one row to another) doesn't change the value of the determinant.

7. Triangle Matrix: The determinant of an upper or lower triangular matrix (all entries above or below the main diagonal are zero) is the product of its diagonal entries.

8. Matrix Inversion: A square matrix is invertible (has an inverse) if and only if its determinant is non-zero.
9. Product of Matrices: The determinant of the product of two square matrices A and B is equal to the product of their determinants: det(AB) = det(A) * det(B).
10. Transpose: The determinant of a matrix is the same as the determinant of its transpose: det(A) = det(A^T).
11. Identity Matrix: The determinant of the identity matrix I is equal to 1: det(I) = 1.
12. Zero Matrix: The determinant of a matrix with all elements equal to zero is 0.

13. Linearity: The determinant is a linear function of each row or column when the other rows or columns are held fixed.

14. Cofactor Expansion: You can compute the determinant of a matrix by expanding along any row or column using cofactors.

15. Adjoint and Inverse: The adjoint of a matrix A is the transpose of the matrix of cofactors of A. The inverse of A is related to its determinant and adjoint: A^(-1) = (1/det(A)) * adj(A).

Question 1:

Find the Value of x if

$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Solution:

Given that,

$\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$2 - 20 = 2x^2 - 24$

Now, bring the x term on the L.H.S

$2x^2 = 6$

$x^2 = 6/2$

To remove the square root on the L.H.S, take the square root on both the sides, then we get

$x = \pm\sqrt{3}$

Thus, the value of x is $\pm\sqrt{3}$

Question 2:

Prove that, $det\begin{pmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{pmatrix} = 4a^2b^2c^2$

Solution:

To Prove: $det\begin{pmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{pmatrix} = 4a^2b^2c^2$

Take, L.H.S:

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Now, take the variables a, b, c from row 1, row 2 and row 3 respectively,

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Now take "a" as common form c_1, "b" form c_2, "c" as common form c_3, then we get

$$= abc\ (abc)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Perform the column operation, $c_2 \to c_2 + c_1$

$$= (abc)^2\begin{vmatrix} -1 & 1-1 & 1 \\ 1 & -1+1 & 1 \\ 1 & 1+1 & -1 \end{vmatrix}$$

$$= (abc)^2\begin{vmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

Again, do the column operation: $c_3 \to c_3 + c_1$

$$= (abc)^2\begin{vmatrix} -1 & 0 & 1-1 \\ 1 & 0 & 1+1 \\ 1 & 2 & -1+1 \end{vmatrix}$$

$$= (abc)^2 \begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= (abc)^2 \left(-1 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}\right)$$

$$= (abc)^2 \left(-1 \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} - 0 + 0\right)$$

$$= (abc)^2 \,(-1(0(0) - 2(2)))$$

$$= (abc)^2 \,(4)$$

$$= 4\,(abc)^2$$

$$= 4a^2b^2c^2$$

Thus, L.H.S = R.H.S

Hence, proved.

Question 3:

By using the properties of determinants, show that: $det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & b^2\end{pmatrix} = (a-b)(b-c)(c-a)$

Solution:

To prove: $det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & b^2\end{pmatrix} = (a-b)(b-c)(c-a)$

Now, take L. H. S

$$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

Now, applying $R_1 \to R_1 - R_2$

$$= \begin{vmatrix}\mathbf{1-1} & a-b & a^2-b^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$= \begin{vmatrix}\mathbf{0} & (a-b) & (a-b)(a+b)\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

$$= \begin{vmatrix}0(\mathbf{a-b}) & (\mathbf{a-b}) & (\mathbf{a}-\boldsymbol{b})(a+b)\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

Taking (a-b) outside from R_1, we get:

$$= (\mathbf{a}-\boldsymbol{b})\begin{vmatrix}0 & 1 & a+b\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}$$

Now, again perform the row operation: $R_2 \to R_2 - R_3$

$$= (a-b)\begin{vmatrix}0 & 1 & a+b\\ \mathbf{1-1} & b-c & b^2-c^2\\ 1 & c & c^2\end{vmatrix}$$

$$= (a-b)\begin{vmatrix} 0 & 1 & a+b \\ \mathbf{0} & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

Taking (b-c) outside from R_2, we get:

$$= (a-b)\,(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Now, expand the determinant along the column 1,

$$= (a-b)\,(b-c)\left(0\begin{vmatrix} 1 & b+c \\ c & c^2 \end{vmatrix} - 0\begin{vmatrix} 1 & a+b \\ c & c^2 \end{vmatrix} + 1\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}\right)$$

$$= (a-b)\,(b-c)\left(0 - 0 + 1\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}\right)$$

= (a-b)(b-c) [(b+c)-(a+b)]

= (a-b)(b-c)(b+c-a-b)

= (a-b) (b-c)(c-a)

= R.H.S

Thus, $det\begin{pmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & b^2\end{pmatrix} = (a-b)(b-c)(c-a)$

L.H. S = R.H.S

Hence, it is proved

Question 4:

Determine the value of k, if the area of triangle is 4 square units

The vertices are: (k,0), (4, 0) and (0, 2)

Solution:

We know that, the area of triangle is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}$$

Given that, the area of triangle is 4 square units

We know that, the area is always positive.

A triangle can have both positive and negative signs.

∴ Δ = ± 4.

Now, substitute the values,

$x_1 = k$, $y_1 = 0$, $x_2 = 4$, $y_2 = 0$, $x_3 = 0$ $y_3 = 2$

$$\pm 4 = \frac{1}{2}\begin{vmatrix}k & 0 & 1\\ 4 & 0 & 1\\ 0 & 2 & 1\end{vmatrix}$$

$$\pm 4 = \frac{1}{2}\begin{vmatrix}k & 0 & 1\\ 4 & 0 & 1\\ 0 & 2 & 1\end{vmatrix}$$

$$\pm 4 = \frac{1}{2}\left(k\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix} - 0\begin{vmatrix}4 & 1\\ 0 & 1\end{vmatrix} + 1\begin{vmatrix}4 & 0\\ 0 & 2\end{vmatrix}\right)$$

$\pm 4 = \frac{1}{2}[\, k\,(0-2) - 0\,(4-0) + 1\,(8-0)\,]$

$\pm 4 \times 2 = (k\,(-2) - 0 + 1\,(8))$

$\pm 8 = -2k + 8$

Therefore, we get:

8 = -2k+8 (or)

-8 = -2k+8

Now, solve both the equations, we get:

Solving $8 = -2k + 8$

$8 - 8 = -2k$

$0 = -2k$

$k = \frac{0}{-2} = 0$

Solving $-8 = -2k + 8$

$-8 - 8 = -2k$

$-16 = -2k$

$k = \frac{-16}{-2} = 8$

Hence, the required value of k is either k = 0 or k = 8

Question 5:

If A is an invertible matrix of order 2, then det (A^{-1}) is equal to:

(i)det (A) (ii) 1/det (A) (iii) 1 (iv) 0

Solution:

The correct answer is option **(B)**

Multiplication system of algebraic equation:-

Unit 1: ALGEBRA-1

What is the solution to this system of equations?

$$y = 2x - 9$$
$$y = -x + 3$$

Solution.

To solve by elimination, multiply by a factor such that when the equations are added, one variable is eliminated.

$$\begin{aligned} y &= 2x - 9 \\ 2(y &= -x + 3) \end{aligned} \quad \rightarrow \quad \begin{aligned} y &= 2x - 9 \\ \underline{2y} &\underline{= -2x + 6} \\ 3y &= -3 \\ y &= -1 \end{aligned} \quad \longrightarrow$$

Solve for x.

$$\begin{aligned} y &= 2x - 9 \\ -1 &= 2x - 9 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

Coordinates of the point of intersection: (4, -1)

CONSISTENCY OF EQUATIONS –

1. Homogeneous Equations:

- Definition: A homogeneous equation is a linear equation in which all the terms have the same degree (exponents) and there are no constant terms.
- Example: - Homogeneous: (2x + 3y - 5z = 0) is a homogeneous equation because all terms have degree 1.
  - Non-Homogeneous: 2x + 3y - 5z = 7 is not homogeneous due to the constant term.

2. Non-Homogeneous Equations:

- Definition: A non-homogeneous equation is a linear equation in which the terms have different degrees or it contains constant terms.
- Example: - Non-Homogeneous: 3x + 2y - 5z = 8 is non-homogeneous due to the constant term and varying degrees of x ,y, and z.

3. Consistent Equations: -

Definition: A system of equations is consistent if it has at least one solution, meaning the equations can be satisfied.

- Example: - Consistent: x + 2y = 5 and 2x + 4y = 10have a common solution (e.g., x = 1 and y = 2, making the system consistent.

4. Non-Consistent Equations (Inconsistent): -

Definition: A system of equations is inconsistent if it has no solution, meaning the equations contradict each other and cannot be simultaneously satisfied.

- Example: - Non-Consistent: $x + 2y = 5$ and $2x + 4y = 11$ have no common solution, making the system inconsistent.

CRAMER'S RULE :-

Consider a system of linear equations with n variables x_1, x_2, x_3, …, x_n written in the matrix form AX = B.

Here,

A = Coefficient matrix (must be a square matrix)

X = Column matrix with variables

B = Column matrix with the constants (which are on the right side of the equations)

Now, we have to find the determinants as:

$D = |A|$, Dx_1, Dx_2, Dx_3,…, Dx_n

Here, Dx_i for i = 1, 2, 3,…, n is the same determinant as D such that the column is replaced with B.

Thus,

$x_1 = Dx_1/D$; $x_2 = Dx_2/D$; $x_3 = Dx_3/D$; ….; $x_n = Dx_n/D$ {where D is not equal to 0}

CRAMER'S RULE FOR 2 UNKNOWN VARIABLES :-

Cramer's rule for the 2×2 matrix is applied to solve the system of equations in two variables.

Let us consider two linear equations in two variables.

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Here,

$$Coefficient\ matrix = A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

$$Variable\ matrix = X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$Constant\ matrix = B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$D = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

And

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1 b_2 - c_2 b_1$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1 c_2 - a_2 c_1$$

Therefore,

$x = D_x/D$

$y = D_y/D$

Let us write these two equations in the form of AX

CRAMER'S RULE FOR 3 UNKNOWN VARIABLES :-

Consider:

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Let us write these equations in the form AX = B.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now,

$$D = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

And

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Therefore, $x = D_x/D$, $y = D_y/D$, $z = D_z/D$; $D \neq 0$.

Condition of consistency and inconsistency:-

1. If D is non zero , the system is consistent and has unique solution.
2. If D= 0 and at leastone of D1,D2,D3 is non zero, the system is inconsistent.
3. If D=0 and D1=D2=D3=0 the system will have either infinite solution or no solution.

Q. Solve the following systems of linear equations by Cramer's rule:

(i) $5x - 2y + 16 = 0, x + 3y - 7 = 0$

(ii) $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

(iii) $3x + 3y - z = 11,\ 2x - y + 2z = 9,\ 4x + 3y + 2z = 25$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

SOLUTION

(i) $5x - 2y + 16 = 0 \ , \ x + 3y - 7 = 0$

The given equations are

$$5x - 2y = -16 \quad \text{-----}(1)$$

$$x + 3y = 7 \quad \text{-----}(2)$$

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix}$$

$$= 15 + 2 = 17 \neq 0$$

$$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix}$$

$$= -48 + 14 = -34$$

$$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix}$$

$$= 35 + 16 = 51$$

By Cramer's rule we get

$$x = \frac{\Delta_1}{\Delta} = -\frac{34}{17} = -2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$$

$\therefore$ The solution is $x = -2 \ , \ y = 3$

(ii) $\frac{3}{x} + 2y = 12 \ , \ \frac{2}{x} + 3y = 13$

Put $\frac{1}{x} = a$ in the above equations.

$$3a + 2y = 12 \quad \text{-----}(1)$$

$$2a + 3y = 13 \quad \text{-----}(2)$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$a = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$$

$\therefore \quad a = \frac{1}{x} = 2 \quad \Rightarrow \quad x = \frac{1}{2}$

$\therefore$ The solution is $x = \frac{1}{2}$, $y = 3$

(iii) $3x + 3y - z = 11$, $2x - y + 2z = 9$, $4x + 3y + 2z = 25$

The given equations are

$3x + 3y - z = 11$ -----(1)

$2x - y + 2z = 9$ -----(2)

$4x + 3y + 2z = 25$ -----(3)

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$= 3(-2-6) - 3(4-8) - 1(6+4)$

$= 3 \times -8 - 3 \times -4 - 1 \times 10$

$\Delta = -24 + 12 - 10 = -22$

$$\Delta_1 = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$= 11(-2-6) - 3(18-50) - 1(27+25)$

$= -88 + 96 - 52 = 96 - 140$

$\Delta_1 = -44$

$$\Delta_2 = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$= 3(18-50) - 11(4-8) - 1(50-36)$

$= 3 \times -32 - 11 \times -4 - 1 \times 14$

$= -96 + 44 - 14$

$\Delta_2 = -66$

$$\Delta_3 = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$= 3[(-25-27) - 3(50-36) + 11(6+4)]$

$$= 3 \times -52 - 3 \times 14 + 11 \times 10$$

$$\Delta_3 = -156 - 42 + 110 = -88$$

$$x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$$

$$z = \frac{-88}{-22} = 4$$

$\therefore$ The solution is $x = 2$, $y = 3$, $z = 4$.

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0$, $\frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$,

$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

The given equations are

$$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} = 1$$

$$\frac{1}{x} + \frac{2}{y} + \frac{1}{z} = 2$$

$$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} = -1$$

Put $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$

$\therefore$

$$3a - 4b - 2c = 1 \quad \text{-----(1)}$$

$$a + 2b + c = 2 \quad \text{-----(2)}$$

$$2a - 5b - 4c = -1 \quad \text{-----(3)}$$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix}$$

$$= 3(-8+5) + 4(-4-2) - 2(-5-4)$$

$$= 3 \times -3 + 4 \times -6 - 2 - 9$$

$$= -9 - 24 + 18$$

$$= -33 + 18 = -15$$

$$\Delta_1 = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$$= 1(-8+5) + 4(-8+1) - 2(-10+2)$$

$$= 1 \times -3 + 4 \times -7 - 2 \times -8$$

$$\Delta_1 = -3 - 28 + 16$$

$$= -31 + 16 = -15$$

$$\Delta_2 = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= 3(-8+1) - 1(-4-2) - 2(-1-4)$$

$$= 3 \times -7 - 1 \times -6 - 2 \times -5$$

$$= -21 + 6 + 10$$

$$\Delta_2 = -21 + 16 = -5$$

$$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$$

$$= 3(-2+10) + 4(-1-4) + 1(-5-4)$$

$$= 3 \times 8 + 4 \times -5 + 1 \times -9$$

$$= 24 - 20 - 9$$

$$\Delta_3 = 24 - 29 = -5$$

$$a = \frac{\Delta_1}{\Delta} = \frac{-15}{-15} = 1$$

$$b = \frac{\Delta_2}{\Delta} = \frac{-5}{-15} = \frac{1}{3}$$

$$c = \frac{\Delta_3}{\Delta} = \frac{13}{-15} = \frac{-5}{-15} = \frac{1}{3}$$

$$a = \frac{1}{x} = 1 \Rightarrow x = 1$$

$$b = \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3$$

$$c = \frac{1}{z} = \frac{1}{3} \Rightarrow z = 3$$

$\therefore$ The solutions of the given system equations are

$$x = 1 \;,\; y = 3 \;,\; z = 3$$